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SOME PROPOSITIONS ABOUT INVERSE SEMIGROUPS

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Abstract. The inverse semigroups are semigroups studied by many algebraists. In this paper we will formulate and prove some other propositions on these semigroups. So we will prove two propositions concerning the closure of a subsemigroup of a given inverse semigroup S, within the meaning introduced by Schein in 1962, two propositions on the group congruence on a normal subsemigroup of the inverse semigroup S, and a proposition about closed subsemigroup assertion of an inverse semigroup S.

Keywords: closure of the semigroup, group congruence, closed semigroup

Some definitons

Definition 1. [1] ***A subsemigroup of the given semigroup*** S ***is called*** *full* ***subsemigroup if it contains all the idempotents of this semigroup*** S***.***

In 1952, Vegner [5] introduced a *natural partial order* on an inverse semigroup S as follows:

$$a \le b \iff \exists e \in E(S),\ a = eb$$

If H is an arbitrary subset of an inverse semigroup *S*, then, Schein in [3] and Clifford and Preston in [4], give this:

Definition 2. ***The*** *closure* ***of*** H ***is the set*** $H\omega$ ***defined as below:***

$$H\omega = \{x \in S \mid \exists h \in H,\ h \le x\}$$

From this definition we see, immediately, that if H and K are subsets of S than:

$$H \subset H\omega;\ \ H \subset K \Rightarrow H\omega \subset K\omega \ \text{ and } \ (H\omega)\omega = H\omega$$

Definition 3. [1], [2] ***The subset*** H ***of the inverse semigroup*** S ***is called*** *closed* ***if we have*** $H\omega = H$

Definition 4. [2] ***The subsemigroup*** N ***of the inverse semigroup*** S ***will be called*** *normal* ***if it is full, closed and*** $\forall x \in S,\ \ x'N\,x \subset N$ ***where*** x' ***is an inverse element of*** x***.***

Definition 5. [2], [6], [7], [8] ***A congruence*** ρ ***on an inverse semigroup*** S ***will be called a*** *group congruence* ***if the factor semigroup*** S/ρ ***is a group.***

Now, we must prove this propositions:

Proposition 1. ***If*** S ***is an inverse semigroup and*** E ***is the set of its idempotents then*** $E\omega$ ***is an normal inverse subsemigroup of*** S***.***

Proof. First we see that:

$$(x \in E\omega) \Rightarrow (\exists e \in E, e \leq x) \Rightarrow (\exists e_1 \in E, e = e_1 x) \Rightarrow (e_1 xx' = ex') \Rightarrow$$
$$(e_1 e_2 = ex') \Rightarrow (e' = ex') \Rightarrow (e' \leq x') \Rightarrow (x' \in E\omega)$$

where $e_1, e_2, e' \in E$, $e_2 = xx'$, $e_1 e_2 = e'$ and x' is an inverse element of x. So, we have shown that $(x \in E\omega) \Rightarrow (x' \in E\omega)$, i.e $E\omega$ is inversive subsemigroup of *S*. Second, $E\omega$ is also full inverse subsemigroup of *S*, because $E \subset E\omega$, that means $E\omega$ contains all idempotents of *S*. Third, $E\omega$ is closed subsemigroup of *S*, because $(E\omega)\omega = E\omega$. Finally, we must shown that $\forall x \in S,\ x' E\omega\, x \subset E\omega$. Indeed we have:

$$(y \in x' E\omega\, x) \Rightarrow (\exists z \in E\omega,\ y = x' zx) \quad \text{and}$$

$$(z \in E\omega) \Rightarrow (\exists e \in E,\ e \leq z) \Rightarrow (\exists f \in E,\ e = fz) \Rightarrow [(xx')e = (xx')(fz)]$$
$$\Rightarrow [(xx')e = f(xx')z] \Rightarrow [(xx')e = (fx)(x'z)] \Rightarrow [(xx')ex = (fx)(x'zx)]$$
$$\Rightarrow [e(xx')x = (fx)y] \Rightarrow [ex = (fx)y] \Rightarrow [x'ex = (x'fx)y] \quad (1)$$

moreover, the elements $x'ex = e',\ x'fx = f'$ are idempotents (Indeed $(x'ex)^2 = x'exx'ex = \qquad = x'xx'eex = x'ex$).

Now, from the last equlity (2) we have $e' = f'y$ and $(e', f' \in E) \Rightarrow (e' \leq y) \Rightarrow (y \in E\omega)$. So, we have proved that $E\omega$ is normal inversive subsemigroup of the semigroup *S*. ■

Proposition 2. ***If*** N ***is an inverse normal subsemigroup of the inverse semigroup of*** S***, then we have:*** $\forall x \in S,\ (Nx)\omega = (xN)\omega$

Proof. Let be $y \in (Nx)\omega$ then we see that:

$$[y \in (Nx)\omega] \Rightarrow (\exists nx \in Nx,\ nx \leq y) \Rightarrow (\exists e \in E, nx = ey) \Rightarrow (x'nx = x'ey) \quad (1)$$

where x' is an inverse element of x. But N is normal, that means $x'Nx \subset N$, so $x'nx = n_1$ and $n_1 \in N$. From the last equality of (1) we will have:

$$(n_1 = x'ey) \Rightarrow (xn_1 = (xx')ey) \Rightarrow (n_1 = e'y)$$

where $e' = (xx')e \in E$ as a product of two idempotents of *E*. Now we can write:

$$(xn_1 = e'y) \Rightarrow (xn_1 \leq y) \Rightarrow [y \in (xN)\omega]$$

Finally, we have prove that $\forall x \in S,\ (Nx)\omega \subset (xN)\omega$. It's the same to prove also $\forall x \in S,\ (Nx)\omega \supset (xN)\omega$, so $\forall x \in S,\ (Nx)\omega = (xN)\omega$ ■

Proposition 3. ***If*** *N* ***is an inverse normal subsemigroup of the inverse semigroup of*** *S*, ***then*** $\rho_N = \{(x, y) \in S^2 \mid xy' \in N\}$ ***is a group congruence on*** *S*, ***and the*** ρ_N ***- class*** N ***is the identity of the factor group*** S / ρ_N.

Proof. First, we must show that the relation ρ_N is an equivalence relation on *S*. a) $\forall x \in S,\ xx'$ is an idempotent, so $xx' \in E$, where *E* is the set of idempotents of *S*. Since *N* is normal, then it will be full. Thus $E \subset N$ that means $xx' \in N$ or we have $x\rho_N x$. We have shown that ρ_N is reflective relation.

b) $\forall (x, y) \in S^2,$ we see:

$$(x\rho_N y) \Rightarrow (xy' \in N) \Rightarrow [(xy')' \in N] \Rightarrow (yx' \in N) \Rightarrow (y\rho_N x)$$

i.e. ρ_N is symetric relation.

c) $\forall (x, y) \in S^2,$ we have also:

$$(x\rho_N y \wedge y\rho_N z) \Rightarrow (xy' \in N \wedge yz' \in N) \Rightarrow [(xy')(yz') \in N] \Rightarrow [(xx'xy')(yz') \in N] \Rightarrow$$
$$[x(x'x)(y'y)z') \in N] \Rightarrow [x(y'y)(x'x)z') \in N] \Rightarrow [(xy')(yx')(xz') \in N] \Rightarrow$$
$$[(xy')(xy')'(xz') \in N] \Rightarrow [(aa')(xz') \in N]$$

where $a = xy'$ and $aa' = e$ is an idempotent in *S*, such that $e(xz') \in N$. Now, if $e(xz') = n$, than $n \le xz'$ that means $xz' \in N\omega$ or $xz' \in N \Rightarrow x\rho_N z$ or ρ_N is a transitive relation, (*N* is normal i.e. *N* is closed, so $N\omega = N$). Thus we have prove that ρ_N is an equivalence relation.

Second, we must show that ρ_N is a congruence. Let we have $x\rho_N y$ and let show now that $\forall c \in S,\ (xc)\rho_N(yc)$ or $(xc)(yc)' \in N$. Indeed,

$$(xc)(yc)' = (xc)(c'y') = (xc)(c'y') = (xx'x)(cc')y' = x(x'x)(cc')y' = x(cc')(x'x)y' =$$
$$(xc)(c'x')(xy') = (xc)(cx)'(xy') = (aa')(xy') = e(xy')$$

where $a = xc$, and $e = aa'$ is an idempotent of *S*. It follows that $e \in N$, because *N* is full, that means $E \subset N$. Now we can write:

$$(e \in N \wedge xy' \in N) \Rightarrow [e(xy') \in N] \Rightarrow [(xc)(yc)' \in N] \Rightarrow (xc)\rho_N(yc)$$

so ρ_N is a right congruence. We can see also that $x\rho_N y \Rightarrow xy' \in N$ and we will need to show $(cx)\rho_N(cy)$. Indeed,

$$(cx)(cy)' = (cx)(y'c') = c(xy')c' = (c')'(xy')c' = a'(xy')a \in a'Na \subset N$$

because N is normal. Thus, $(cx)(cy)' \in N$, that means $(cx)\rho_N(cy)$ and so we conclude that ρ_N is also left congruence, i.e. congruence on *S*.

Third, we must to show that N is a ρ_N - class:

a) $\forall(x,y)\in N^2,\ (x\in N \wedge y'\in N) \Rightarrow (\mathrm{xy'} \in \mathrm{N}) \Rightarrow x\rho_N y$ so, all the elements of N belongs in the same ρ_N - class.

b) $\forall n\in N,\ \forall x\in S,\ n\rho_N x \Rightarrow nx'\in N$. If $nx'=n_1$ we have $(n_1 = nx') \Rightarrow [n_2 = n'n_1 = (n'n)x' = ex]$ where $n_2 = n'n_1 \in N$ and $n'n = e\in E\subset N$, so we will have: $(n_2 = ex') \Rightarrow (n_2 \le x') \Rightarrow (x'\in N\omega) \Rightarrow (x'\in N) \Rightarrow (x\in N)$. Now, from a) and b), we can conclude that N is a ρ_N - class.

Since $E\subset N$, i.e. all the idempotents belong in the same ρ_N - class, *N*, it follows that ρ_N is a group congruence and is clear that N is the unite element of the group S/ρ_N. Indeed, fo any class $\overline{x}\in S/\rho_N$ we have we have $\overline{x}N = \overline{x}\cdot\overline{e} = \overline{xe} = \overline{x}$ because we have $xe\rho_N x$ and $xex'\in E$ Indeed, $(xex')^2 = xe(x'x)ex' = x(x'x)eex' = xex'$, so $xex'\in E\subset N$ that means $xe\rho_N x'$ or $\overline{xe} = \overline{x}$. It is the same to show $N\overline{x} = \overline{x}$. ■

Proposition 4. ***If*** τ ***is a group congruence on*** *S*, ***than exists an inverse normal subsemigroup*** N ***of*** *S* ***such that*** $\tau = \rho_N$.

Proof. Since τ is a group congruence on *S*, it follows that S/τ is a group. Let be N the τ - class that is the identity element of the group S/τ. Now we will prove that $\tau = \rho_N$.

First, N is a subsemigroup of S because:

$$(x\in N \wedge y\in N) \Rightarrow (\overline{x} = N \wedge \overline{y} = N) \Rightarrow (\overline{x}\cdot\overline{y} = N^2 = N) \Rightarrow (\overline{xy} = N) \Rightarrow (xy\in N)$$

Second, N is the identity element of the group S/ρ_N that means all the idempotents of S belongs to *N*, i.e. N is full, so $\forall x\in N,\ xx' = e\in E\subset N$. On the other hand, we have:

$$(xx' = e\in N) \Rightarrow (\overline{xx'} = \overline{e} = N) \Rightarrow (\overline{x}\cdot\overline{x'} = N) \Rightarrow (N\overline{x'} = N) \Rightarrow (\overline{x'} = N) \Rightarrow (x'\in N)$$

Where $\overline{x} = N$, because $x\in N$. Thus N is inversive subsemigroup of *S*.

Third, N is closed subsemigroup os *S*. Indeed,

$$(x \in N\omega) \Rightarrow (\exists n \in N,\ n \le x) \Rightarrow (\exists e \in E \subset N,\ n = ex)$$
$$\Rightarrow (N = \overline{n} = \overline{ex} = \overline{e} \cdot \overline{x} = N\overline{x} = \overline{x})$$

So, $\overline{x} = N$ that means $x \in N$ and $N\omega = N$ (the other inclusion $N\omega \supset N$ is evident from the definition of closure of N), i.e. N is closed.

Fourth, we need to show that $\forall x \in S,\ x'Nx \subset N$. Indeed,

$$(y \in x'Nx) \Rightarrow (y = x'nx,\ n \in N) \Rightarrow (\overline{y} = \overline{x'nx}) \Rightarrow$$
$$(\overline{y} = \overline{x'} \cdot \overline{n} \cdot \overline{x} = \overline{x'} \cdot N \cdot \overline{x}) \Rightarrow (\overline{y} = \overline{x'} \cdot \overline{x}) \Rightarrow (\overline{y} = \overline{x'x} = \overline{e} = N)$$

So, $\overline{y} = N$ that means $y \in N$, i.e. N is normal and identity element for both grupet S/τ and S/ρ_N

Fifth, it remains to be shown that $\tau = \rho_N$, where $\rho_N = \{(x,y) \in S^2 \mid xy' \in N\}$, since τ is congruence, we have:

$$(x\tau y) \Rightarrow [(xy')\tau(yy')] \Rightarrow [(xy')\tau e] \Rightarrow [\overline{(xy')}^{\tau} = \overline{e}^{\tau} = N] \Rightarrow (xy' \in N) \Rightarrow (x\rho_N y)$$

Thus, we have prove that $\tau \subset \rho_N$ (2), vice versa we have:

$$(x\rho_N y) \Rightarrow (xy' \in N) \Rightarrow (\overline{xy'}^{\rho_N} = N) \Rightarrow (\overline{xy'}^{\rho_N} = \overline{yy'}^{\rho_N} = \overline{e}^{\rho_N} = N) \Rightarrow$$
$$[(xy' \in N) \wedge (yy' \in N)] \Rightarrow [(xy')\tau(yy')] \Rightarrow [(xy'y)\tau(yy'y)] \Rightarrow$$
$$[(xe_1)\tau y] \Rightarrow (\overline{xe_1}^{\tau} = \overline{y}^{\tau}) \Rightarrow (\overline{x}^{\tau}N = \overline{y}^{\tau}) \Rightarrow (\overline{x}^{\tau} = \overline{y}^{\tau}) \Rightarrow (x\tau y)$$

So, it follows that we have also $\rho_N \subset \tau$ (3)

Now, from (2) and (3) we can conclude that $\tau = \rho_N$ ■

Proposition 5. ***If U is a closed inverse subsemigroup of T where T is a closed inverse subsemigroup of the semigroup S, then U is a closed inverse subsemigroup of S***

Proof. Let be $U\omega_T$ the closure of U with respect to *T*, $U\omega_S$ the closure of U with respect to *S* and $T\omega_S$ the closure of T with respect to *S*. So, we have: $U\omega_T = U$ and $T\omega_S = T$

Now, we must show that $U\omega_S = U$ First, we see that $U\omega_T \subset U\omega_S$, because,

$$(x \in U\omega_T) \Rightarrow [\exists u \in U,\ (u \le x) \wedge (x \in T)] \Rightarrow [\exists u \in U,\ (u \le x) \wedge (x \in S)] \Rightarrow (u \in U\omega_S)$$

Second, we see: $(x \in U\omega_S) \Rightarrow (x \in T\omega_S)$ because: $(U \subset T) \Rightarrow (U\omega_S \subset T\omega_S = T)$, *i.e.* $x \in T$ further $U\omega_S \subset T\omega_S$, and now we can write

$$(x \in U\omega_S) \Rightarrow (x \in T\omega_S) \Rightarrow [\exists u \in U,\ (u \leq x) \wedge (x \in T)] \Rightarrow (x \in U\omega_T = U) \Rightarrow (x \in U)$$

Finally, we have proved that $U\omega_S = U$, so, U is a closed inverse subsemigroup of *S*. ■

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