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Generalization of strong convergence theorem in CAT(0) spaces

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Abstract. The aim of this paper is to give the generalization condition of T-Ciric quasi contractive mapping. Also to study the generalization of strong convergence theorem of modified S-iteration process for Ciric quasi contractive operator in the framework of CAT(0) spaces based on new generalized condition for T-Ciric quasi contractive mapping. Our results extend and generalize many known results from the previous work given in the existing literature (see [1,6]).

1. Introduction and Preliminaries

CAT(0)space. A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces (see [3]), R-trees (see [11]), Euclidean buildings (see [12]), the complex Hilbert ball with a hyperbolic metric (see [13]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [3]. Fixed point theory in CAT(0) spaces was first studied by Kirk (see [1,2]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then, the fixed point theory for single-valued and multi-valued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared.
Let (X, d) be a metric space. A geodesic path joining $x, y \in X$ x ∈ X is a map $c : [0, d(x, y)] \to X$ such that:

- $c(0) = x$
- $c\big(d(x, y)\big) = y$
- d (c (t_1), c (t_2)) = $|t_1 - t_2|$,$\forall\, t_1, t_2$ ∈[0, $d(x, y)$].

The image α of c is called a geodesic (or metric) segment joining x and y. We say X is (i) a geodesic space if any two points of X are joined by a geodesic and (ii) uniquely geodesic if there is exactly one geodesic joining x and y for each x, y ∈ X, which we will denote by [x, y], called the segment joining x to y.

Comparision triangle
A geodesic triangle $\Delta(p,q,r)$. in a geodesic metric space (X, d) consists of three points in $p,q,r \in X$ and a geodesic segment between each pair of vertices $[p,q],[q,r],[r,p]$.
A comparison triangle for the geodesic triangle $\Delta(p,q,r)$ in (X, d) is a triangle $\bar{\Delta}\,(\bar{p},\bar{q},\bar{r}) \subset \mathbb{R}^2$ such that:

- $d(p,q) = d\,(\bar{p},\bar{q})$
- $d(q,r) = d\,(\bar{q},\bar{r})$
- $d(r,p) = d\,(\bar{r},\bar{p})$

Definition of CAT(0) space
Let (X,d) be a geodesic metric space. It is called *CAT(0) space* if for any geodesic triangle $\Delta \in X$ and $x,y \in \Delta$:
$d(x,y) \le \mathrm{d}\,(\bar{x},\bar{y})$ ku $\bar{x},\bar{y} \in \bar{\Delta}$

2. Main Result

2.1. Generalization of T-Ciric Quasi Contraction Mapping

Let X be a CAT(0) space and $S,T : X \to X$ be two mappings. Then S is called *T*-Ciric quasi contraction mapping if it satisfies the following condition:
(1.1)

$$d(TSx,TSy) \le h\max\left\{d(Tx,Ty),\frac{d(Tx,TSx)+d(Ty,TSy)}{2},\frac{d(Tx,TSy)+d(Ty,TSx)}{2}\right\}$$

$(TCQC)$

for all $x,y \in X$ and $0 < h < 1.$
Then the condition (TCQC) can be generalized as follows:
(4.18)

$$d(TSx,TSy) \le h\max\left\{d(Tx,Ty),\frac{d(Tx,TSx)+d(Ty,TSy)}{m},\frac{d(Tx,TSy)+d(Ty,TSx)}{m}\right\}$$

$(TCQC)^*$

for all $x,y \in X$ and $0 < h < \frac{m}{2}.$

2.2. Proof

Each of the conditions $(TZ_1)-(TZ_3)$ implies $(TCQC)^*$

(TZ_1) $d(TSx,TSy)\le ad(Tx,Ty)\le a\frac{m}{2}d(Tx,Ty),\quad 0<a<1,\ m\ge 2.$

(TZ_2) $d(TSx,TSy)\le b\left[d(Tx,TSx)+d(Ty,TSy)\right],\quad 0<b<\frac{1}{2}$

(TZ_3) $d(TSx,TSy)\le c\left[d(Tx,TSy)+d(Ty,TSx)\right],\quad 0<c<\frac{1}{2}$

implies:

$$d(TSx,TSy)\le \max\left\{a\frac{m}{2}d(Tx,Ty), bm\frac{d(Tx,TSx)+d(Ty,TSy)}{m}, cm\frac{d(Tx,TSy)+d(Ty,TSx)}{m}\right\}$$

$$\le h\max\left\{d(Tx,Ty),\frac{d(Tx,TSx)+d(Ty,TSy)}{m},\frac{d(Tx,TSy)+d(Ty,TSx)}{m}\right\}$$

when $h=\max\left\{a\frac{m}{2}, bm, cm\right\}$.

$$0<a<1\Rightarrow 0<a\frac{m}{2}<\frac{m}{2}$$

$$0<b<\frac{1}{2}\Rightarrow 0<bm<\frac{m}{2}\qquad\Rightarrow 0<h<\frac{m}{2}.$$

$$0<c<\frac{1}{2}\Rightarrow 0<cm<\frac{m}{2}$$

3. Generalization of strong convergence theorems in CAT(0) spaces

3.1. Theorem

Let C be a nonempty closed convex subset of a complete CAT(0) space. Let $S,T:C\to C$ *be two commuting mappings such that T is continuous, one-to-one, sub-sequentially convergent and* $S:C\to C$ *is a T-Ciric quasi-contractive operator satisfying (TCQC)** *with* $0 < h < \frac{m}{2}, m \geq 2.$ *Let* $\{x_n\}$ *be defined by the iteration scheme (1.8) [1] . If* $\sum_{n=1}^{\infty}\alpha_n = \infty,\ \sum_{n=1}^{\infty}\alpha_n\beta_n = \infty,\ \sum_{n=1}^{\infty}\alpha_n\beta_n\gamma_n = \infty,$ *then* $\{Tx_n\}$ *converges strongly to Tu, where u is the fixed point of the operator S in C.*

3.2. Proof

From Theorem 1.1 [1], we get that *S* has a unique fixed point in *C*, say *u*. Consider $x, y \in C$. Since S in a T-Ciric quasi-contractive operator satisfying (TCQC)*, then if

$$d(TSx,TSy) \leq \frac{h}{m}\left[d(Tx,TSx)+d(Ty,TSy)\right]$$
$$\leq \frac{h}{m}\left[d(Tx,TSx)+d(Ty,Tx)+d(Tx,TSx)+d(TSx,TSy)\right],$$

Implies

$$\left(1-\frac{h}{m}\right)d(TSx,TSy) \leq \frac{h}{m}d(Tx,Ty)+\frac{2h}{m}d(Tx,TSx),$$

Which yields (using the fact that $0 < h < \frac{m}{2}, m \geq 2)$

$$d(TSx,TSy) \leq \left(\frac{h/m}{1-h/m}\right)d(Tx,Ty)+\left(\frac{2h/m}{1-h/m}\right)d(Tx,TSx).$$

If

$$d(TSx,TSy) \leq \frac{h}{m}\left[d(Tx,TSy)+d(Ty,TSx)\right]$$
$$\leq \frac{h}{m}\left[d(Tx,TSx)+d(TSx,TSy)+d(Ty,Tx)+d(Tx,TSx)\right]$$

Implies

$$\left(1-\frac{h}{m}\right)d(TSx,TSy) \leq \frac{h}{m}d(Tx,Ty)+\frac{2h}{m}d(Tx,TSx)$$

Which also yields (using the fact that $0 < h < \frac{h}{m}, m \geq 2)$

(4.9) $$d(TSx,TSy) \le \left(\frac{h/m}{1-h/m}\right)d(Tx,Ty)+\left(\frac{2h/m}{1-h/m}\right)d(Tx,TSx).$$

Denote

$$\delta = \max\left\{h, \frac{h/m}{1-h/m}\right\} = h,$$

$$L = \frac{2h/m}{1-h/m}.$$

Thus,in all cases,

$$d(TSx,TSy) \le \delta d(Tx,Ty) + Ld(Tx,TSx)$$

(4.20) $$= hd(Tx,Ty)+\left(\frac{2h/m}{1-h/m}\right)d(Tx,TSx).$$

holds for all $x, y \in C$.

Also from (TCQC)* with $y = u = Su$, we have

$$d(TSx,TSu) \le h\max\left\{d(Tx,Tu), \frac{d(Tx,TSx)}{m}, \frac{d(Tx,TSu)+d(Tu,TSx)}{m}\right\}$$

$$\le h\max\left\{d(Tx,Tu), \frac{d(Tx,Tu)+d(Tu,TSx)}{m}, \frac{d(Tx,TSu)+d(Tu,TSx)}{m}\right\}$$

$$= h\max\left\{d(Tx,Tu), \frac{d(Tx,Tu)+d(Tu,TSx)}{m}\right\}$$

(4.21) $$\le hd(Tx,Tu).$$

Now (4.21) gives

(4.22) $$d(TSx_n,Tu) \le hd(Tx_n,Tu).$$

(4.23) $$d(TSy_n,Tu) \le hd(Ty_n,Tu).$$

(4.24) $$d(TSz_n,Tu) \le hd(Tz_n,Tu).$$

Using (1.8),(2.6) and Lemma 1.1(ii) [1], we have

$$d(Tz_n,Tu) = d(\gamma_n TSx_n \oplus (1-\gamma_n)Tx_n, Tu)$$

$$\le \gamma_n d(TSx_n, Tu) + (1-\gamma_n) d(Tx_n, Tu)$$

(4.25) $$\le \gamma_n h d(Tx_n Tu) + (1-\gamma_n) d(Tx_n, Tu)$$

$$\le [1-(1-h)\gamma_n] d(Tx_n, Tu).$$

Again using (1.8),(2.5),(2.7) and Lemma 1.1(ii) [1], we have

$$d(Ty_n, Tu) \le d(\beta_n TSz_n \oplus (1-\beta_n) Tx_n, Tu)$$

$$\le \beta_n d(TSz_n, u) + (1-\beta_n) d(Tx_n, Tu)$$

$$\le \beta_n h d(Tz_n Tu) + (1-\beta_n) d(Tx_n, Tu)$$

(4.26) $$\le \beta_n h[1-(1-h)\gamma_n] d(Tx_n Tu) + (1-\beta_n) d(Tx_n, Tu)$$

$$\le [1-(1-h)\beta_n - h(1-h)\beta_n\gamma_n] d(Tx_n, Tu).$$

Now using (1.8),(2.4),(2.8), $TS = ST$ (by assumption of the theorem) and Lemma 1.7(ii) [1], we have

$$d(Tx_{n+1}, Tu) = d(\alpha_n STy_n \oplus (1-\alpha_n) Tx_n, Tu)$$

$$\le \alpha_n d(STy_n, Tu) + (1-\alpha_n) d(Tx_n, Tu)$$

$$\le \alpha_n h d(Ty_n, Tu) + (1-\alpha_n) d(Tx_n, Tu)$$

$$\le \alpha_n h[1-(1-h)\beta_n - h(1-h)\beta_n\gamma_n] d(Tx_n, Tu) + (1-\alpha_n) d(Tx_n, Tu)$$

$$\le [1-\{(1-h)\alpha_n - h(1-h)\beta_n\gamma_n + h^2(1-h)\alpha_n\beta_n\gamma_n\}] d(Tx_n, Tu)$$

$$= (1-\beta_n) d(Tx_n, Tu),$$

Where $\beta_n = \{(1-h)\alpha_n - h(1-h)\alpha_n\beta_n + h^2(1-h)\alpha_n\beta_n\gamma_n\},$ since $0 < h < \frac{m}{2}, m \ge 2,$ and by assumption of the theorem $\sum_{n=1}^{\infty} \alpha_n = \infty,$ $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty,$ $\sum_{n=1}^{\infty} \alpha_n\beta_n\gamma_n = \infty,$ *it follows that* $\sum_{n=1}^{\infty} \beta_n = \infty,$ therefore by Lemma 1.8 [1], we get that $\lim_{n\to\infty} d(Tx_n, Tu) = 0.$ Therefore $\{Tx_n\}$ converges strongly to $Tu,$ where *u* is the fixed point of the operator *S* in *C*. This completes the proof.

3.3. Corollary 1

Let C be a nonempty closed convex subset of a complete CAT(0) space. Let $S, T : C \to C$ *be two commuting mappings such that T is continuous, one-to-one, subsequentially convergent and* $S : C \to C$ *is T-Kannan contractive operator satisfying the condition*

$$d(TSx, TSy) \leq b\left[\frac{d(Tx, TSx) + d(Ty, TSy)}{m}\right],$$

$$\forall x, y \in X; b \in \left(0, \frac{1}{m}\right), \forall m \geq 2.$$

Let $\{Tx_n\}$ *be defined by the iteration scheme (1.8) [1]. If* $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$, *and* $\sum_{n=1}^{\infty} \alpha_n \beta_n \gamma_n = \infty$, *then* $\{Tx_n\}$ *converges strongly to Tu, where u is the fixed point of the operator S in C.*

3.4. Corollary 2

Let C be a nonempty closed convex subset of a complete CAT(0) space. Let $S, T : C \to C$ *be two commuting mappings such that T is continuous, one-to-one, subsequentially convergent and* $S : C \to C$ *is T-Chatterjea contractive operator satisfying the condition*

$$d(TSx, TSy) \leq c\left[\frac{d(Tx, TSx) + d(Ty, TSy)}{m}\right],$$

$$\forall x, y \in X; c \in \left(0, \frac{1}{m}\right), \forall m \geq 2.$$

Let $\{Tx_n\}$ *be defined by the iteration scheme (1.8) [1]. If* $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$, *and* $\sum_{n=1}^{\infty} \alpha_n \beta_n \gamma_n = \infty$, *then* $\{Tx_n\}$ *converges strongly to Tu, where u is the fixed point of the operator S in C.*

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