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Generalization of strong convergence theorem in CAT(0) spaces

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Abstract. The aim of this paper is to give the generalization condition of T-Ciric quasi contractive mapping. Also to study the generalization of strong convergence theorem of modified S-iteration process for Ciric quasi contractive operator in the framework of CAT(0) spaces based on new generalized condition for T-Ciric quasi contractive mapping. Our results extend and generalize many known results from the previous work given in the existing literature (see [1,6]).

1. Introduction and Preliminaries

CAT(0) space. A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as ‘thin’ as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces (see [3]), R-trees (see [11]), Euclidean buildings (see [12]), the complex Hilbert ball with a hyperbolic metric (see [13]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [3]. Fixed point theory in CAT(0) spaces was first studied by Kirk (see [1,2]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then, the fixed point theory for single-valued and multi-valued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared.

Let \((X, d)\) be a metric space. A geodesic path joining \(x, y \in X\) is a map \(c : [0, d(x, y)] \rightarrow X\) such that:

- \(c(0) = x\)
- \(c(d(x, y)) = y\)
- \(d(c(t_1), c(t_2)) = |t_1 - t_2|\) \(\forall t_1, t_2 \in [0, d(x, y)]\)

The image \(c\) of \(c\) is called a geodesic (or metric) segment joining \(x\) and \(y\). We say \(X\) is (i) a geodesic space if any two points of \(X\) are joined by a geodesic and (ii) uniquely geodesic if there is exactly one geodesic joining \(x\) and \(y\) for each \(x, y \in X\), which we will denote by \([x, y]\), called the segment joining \(x\) to \(y\).

Comparison triangle

A geodesic triangle \(\Delta(p, q, r)\), in a geodesic metric space \((X, d)\) consists of three points in \(p, q, r \in X\) and a geodesic segment between each pair of vertices \([p, q], [q, r], [r, p]\).

A comparison triangle for the geodesic triangle \(\Delta(p, q, r)\) in \((X, d)\) is a triangle \(\tilde{\Delta}(\tilde{p}, \tilde{q}, \tilde{r}) \subset \mathbb{R}^2\) such that:

- \(d(p, q) = d(\tilde{p}, \tilde{q})\)
- \(d(q, r) = d(\tilde{q}, \tilde{r})\)
- \(d(r, p) = d(\tilde{r}, \tilde{p})\)
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Definition of CAT(0) space
Let \((X, d)\) be a geodesic metric space. It is called CAT(0) space if for any geodesic triangle \(\Delta\) in \(X\) and \(x, y \in \Delta\):
\[d(x, y) \leq d(\bar{x}, \bar{y}) \quad \text{for} \quad \bar{x}, \bar{y} \in \Delta\]

Main Result

Generalization of T-Ciric Quasi Contraction Mapping

Let \(X\) be a CAT(0) space and \(S, T : X \to X\) be two mappings. Then \(S\) is called \(T\)-Ciric quasi contraction mapping if it satisfies the following condition:
\[
(1.1) \quad d(TSx, TSy) \leq h \max \left\{ d(Tx, Ty), \frac{d(Tx, TSx) + d(Ty, TSy)}{2}, \frac{d(Tx, TSy) + d(Ty, TSx)}{2} \right\}
\]
\((TCQC)\)
for all \(x, y \in X\) and \(0 < h < 1\).

Then the condition \((TCQC)\) can be generalized as follows:
\[
(4.18) \quad d(TSx, TSy) \leq h \max \left\{ d(Tx, Ty), \frac{d(Tx, TSx) + d(Ty, TSy)}{m}, \frac{d(Tx, TSy) + d(Ty, TSx)}{m} \right\}
\]
\((TCQC)'\)
for all \(x, y \in X\) and \(0 < h < \frac{m}{2}\).

Proof

Each of the conditions \((TZ_1) - (TZ_3)\) implies \((TCQC)'\):
\[
(TZ_1) \quad d(TSx, TSy) \leq ad(Tx, Ty) \leq \frac{m}{2} d(Tx, Ty), \quad 0 < a < 1, \ m \geq 2.
\]
\[
(TZ_2) \quad d(TSx, TSy) \leq b[d(Tx, TSx) + d(Ty, TSy)], \quad 0 < b < \frac{1}{2}
\]
\[
(TZ_3) \quad d(TSx, TSy) \leq c[d(Tx, TSy) + d(Ty, TSx)], \quad 0 < c < \frac{1}{2}
\]
implies:
\[
d(TSx, TSy) \leq \max \left\{ a \frac{m}{2} d(Tx, Ty), bm \frac{d(Tx, TSx) + d(Ty, TSy)}{m}, cm \frac{d(Tx, TSy) + d(Ty, TSx)}{m} \right\}
\]
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Theorem

Let $C$ be a nonempty closed convex subset of a complete CAT(0) space. Let $S, T : C \to C$ be two commuting mappings such that $T$ is continuous, one-to-one, sub-sequentially convergent and $S : C \to C$ is a T-Ciric quasi-contractive operator satisfying (TCQC)’ with $0 < h < \frac{m}{2}$, $m \geq 2$. Let $\{x_n\}$ be defined by the iteration scheme (1.8) [1] . If

$$0 < a < 1 \Rightarrow 0 < a \frac{m}{2} < \frac{m}{2}$$

$$0 < b < \frac{1}{2} \Rightarrow 0 < bm < \frac{m}{2} \Rightarrow 0 < bm < \frac{m}{2}$$

$$0 < c < \frac{1}{2} \Rightarrow 0 < cm < \frac{m}{2}$$

Then

$$\{x_n\}$$

converges strongly to $u$, where $u$ is the fixed point of the operator $S$ in $C$.

Proof

From Theorem 1.1 [1], we get that $S$ has a unique fixed point in $C$, say $a$. Consider $x, y \in C$.

Since $S$ is a T-Ciric quasi-contractive operator satisfying (TCQC)’, then if

$$d(TSx, TSy) \leq \frac{h}{m} \left[d(Tx, TSx) + d(Ty, TSy)\right]$$

$$\leq \frac{h}{m} \left[d(Tx, TSx) + d(Ty, Tx) + d(Tx, TSx) + d(TSx, TSy)\right],$$

Implies

$$\left(1 - \frac{h}{m}\right)d(TSx, TSy) \leq \frac{h}{m} d(Tx, Ty) + \frac{2h}{m} d(Tx, TSx),$$

$$0 < h < \frac{m}{2}, m \geq 2)$$

Which yields (using the fact that
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\[ d(TS_x, TS_y) \leq \left( \frac{h/m}{1-h/m} \right) d(Tx, Ty) + \left( \frac{2h/m}{1-h/m} \right) d(Tx, TS_x). \]

If
\[ d(TS_x, TS_y) \leq \frac{h}{m} \left[ d(Tx, TS_y) + d(Ty, TS_x) \right] \]

\[ \leq \frac{h}{m} \left[ d(Tx, TS_x) + d(TS_x, TS_y) + d(Ty, Tx) + d(Tx, TS_x) \right] \]

Implies
\[ \left( 1 - \frac{h}{m} \right) d(TS_x, TS_y) \leq \frac{h}{m} d(Tx, Ty) + \frac{2h}{m} d(Tx, TS_x) \]

Which also yields (using the fact that \( 0 < h < \frac{h}{m}, m \geq 2 \))
\[ (4.9) \quad d(TS_x, TS_y) \leq \left( \frac{h/m}{1-h/m} \right) d(Tx, Ty) + \left( \frac{2h/m}{1-h/m} \right) d(Tx, TS_x). \]

Denote
\[ \delta = \max \left\{ h, \frac{h/m}{1-h/m} \right\} = h, \]
\[ L = \frac{2h/m}{1-h/m}. \]

Thus, in all cases,
\[ d(TS_x, TS_y) \leq \delta d(Tx, Ty) + Ld(Tx, TS_x) \]
\[ = hd(Tx, Ty) + \left( \frac{2h/m}{1-h/m} \right) d(Tx, TS_x). \]

(4.20)

holds for all \( x, y \in C \).

Also from (TCQC) with \( y = u = Su \), we have
\[ d(TS_x, TS_u) \leq h \max \left\{ d(Tx, Tu), \frac{d(Tx, TS_x)}{m}, \frac{d(Tx, TS_x) + d(Tu, TS_x)}{m} \right\} \]
\[ \leq h \max \left\{ d(Tx, Tu), \frac{d(Tx, Tu) + d(Tu, TS_x)}{m}, \frac{d(Tx, TS_x) + d(Tu, TS_x)}{m} \right\} \]
\[ = h \max \left\{ d(Tx, Tu), \frac{d(Tx, Tu) + d(Tu, TS_x)}{m} \right\} \]
\[ \leq hd(Tx, Tu). \]

(4.21)

Now (4.21) gives
Using (1.8), (2.6) and Lemma 1.1(ii) [1], we have
\[ d(Tz_n, Tu) = d(\gamma_n Tz_n \oplus (1 - \gamma_n)Tx_n, Tu) \]
\[ \leq \gamma_n d(Tz_n, Tu) + (1 - \gamma_n) d(Tx_n, Tu) \]
(4.25)
\[ \leq \gamma_n h d(Tz_n, Tu) + (1 - \gamma_n) d(Tx_n, Tu) \]
\[ \leq \gamma_n [1 - (1 - h) \gamma_n] d(Tx_n, Tu) \]

Again using (1.8), (2.5), (2.7) and Lemma 1.1(ii) [1], we have
\[ d(Ty_n, Tu) \leq d(\beta_n Tz_n \oplus (1 - \beta_n)Tx_n, Tu) \]
\[ \leq \beta_n d(Tz_n, Tu) + (1 - \beta_n) d(Tx_n, Tu) \]
\[ \leq \beta_n h d(Tz_n, Tu) + (1 - \beta_n) d(Tx_n, Tu) \]
\[ \leq \beta_n [1 - (1 - h) \gamma_n] d(Tx_n, Tu) + (1 - \beta_n) d(Tx_n, Tu) \]
\[ \leq [1 - (1 - h) \gamma_n] d(Tx_n, Tu) \]
(4.26)
\[ \leq [1 - (1 - h) \beta_n - h(1 - h) \beta_n \gamma_n] d(Tx_n, Tu) \]

Now using (1.8), (2.4), (2.8), \( TS = ST \) (by assumption of the theorem) and Lemma 1.7(ii) [1], we have
\[ d(Tx_{n+1}, Tu) = d(\alpha_n STy_n \oplus (1 - \alpha_n)Tx_n, Tu) \]
\[ \leq \alpha_n d(STy_n, Tu) + (1 - \alpha_n) d(Tx_n, Tu) \]
\[ \leq \alpha_n h d(Ty_n, Tu) + (1 - \alpha_n) d(Tx_n, Tu) \]
\[ \leq \alpha_n [1 - (1 - h) \beta_n - h(1 - h) \beta_n \gamma_n] d(Tx_n, Tu) + (1 - \alpha_n) d(Tx_n, Tu) \]
\[ \leq [1 - \gamma_n] d(Tx_n, Tu) \]

Where
\[ \beta_n = \{(1 - h) \alpha_n - h(1 - h) \alpha_n \beta_n + h^2(1 - h) \alpha_n \beta_n \gamma_n\} \]

since
\[ 0 < h < \frac{m}{2}, m \geq 2, \] and by assumption of the theorem \( \sum_{n=1}^{\infty} \alpha_n = \infty, \)
\[ \sum_{n=1}^{\infty} \alpha_n \beta_n = \infty, \sum_{n=1}^{\infty} \alpha_n \beta_n \gamma_n = \infty, \text{ it follows that } \sum_{n=1}^{\infty} \beta_n = \infty, \text{ therefore by Lemma 1.8 [1], we get that } \lim_{n \to \infty} d(Tx_n, Tu) = 0. \]
Therefore \( \{Tx_n\} \) converges strongly to \( Tu, \) where \( u \) is the fixed point of the operator \( S \) in \( C. \) This completes the proof. \( \square \)
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Corollary 1
Let $C$ be a nonempty closed convex subset of a complete CAT(0) space. Let $S, T : C \to C$ be two commuting mappings such that $T$ is continuous, one-to-one, subsequentially convergent and $S : C \to C$ is $T$-Kannan contractive operator satisfying the condition

$$d(TSx, TSy) \leq b \left[ \frac{d(Tx, TSx) + d(Ty, TSy)}{m} \right],$$

$\forall x, y \in X; b \in \left(0, \frac{1}{m}\right), \forall m \geq 2.$

Let $\{T_{n}\}$ be defined by the iteration scheme (1.8) [1]. If $\sum_{n=1}^{\infty} \alpha_{n} = \infty$, $\sum_{n=1}^{\infty} \alpha_{n} \beta_{n} = \infty$, and $\sum_{n=1}^{\infty} \alpha_{n} \beta_{n} \gamma_{n} = \infty$, then $\{T_{n}\}$ converges strongly to $Tu$, where $u$ is the fixed point of the operator $S$ in $C$.

Corollary 2
Let $C$ be a nonempty closed convex subset of a complete CAT(0) space. Let $S, T : C \to C$ be two commuting mappings such that $T$ is continuous, one-to-one, subsequentially convergent and $S : C \to C$ is $T$-Chatterjea contractive operator satisfying the condition

$$d(TSx, TSy) \leq c \left[ \frac{d(Tx, TSx) + d(Ty, TSy)}{m} \right],$$

$\forall x, y \in X; c \in \left(0, \frac{1}{m}\right), \forall m \geq 2.$

Let $\{T_{n}\}$ be defined by the iteration scheme (1.8) [1]. If $\sum_{n=1}^{\infty} \alpha_{n} = \infty$, $\sum_{n=1}^{\infty} \alpha_{n} \beta_{n} = \infty$, and $\sum_{n=1}^{\infty} \alpha_{n} \beta_{n} \gamma_{n} = \infty$, then $\{T_{n}\}$ converges strongly to $Tu$, where $u$ is the fixed point of the operator $S$ in $C$.

REFERENCES

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