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Generalization of strong convergence theorem in CAT(0) spaces

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Abstract. The aim of this paper is to give the generalization condition of T-Ciric quasi contractive mapping. Also to study the generalization of strong convergence theorem of modified S-iteration process for Ciric quasi contractive operator in the framework of CAT(0) spaces based on new generalized condition for T-Ciric quasi contractive mapping. Our results extend and generalize many known results from the previous work given in the existing literature (see [1,6]).

1. Introduction and Preliminaries

CAT(0)space. A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces (see [3]), R-trees (see [11]), Euclidean buildings (see [12]), the complex Hilbert ball with a hyperbolic metric (see [13]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [3]. Fixed point theory in CAT(0) spaces was first studied by Kirk (see [1,2]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then, the fixed point theory for single-valued and multi-valued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared.
Let (X, d) be a metric space. A geodesic path joining $x, y \in X$ x ∈ X is a map $c : [0, d(x,y)] \to X$ such that:

- $c(0) = x$
- $c\big(d(x,y)\big) = y$
- d (c (t_1), $c\ (t_2)$) = $|t_1 - t_2|$,$\forall\, t_1, t_2 \in [0, d(x,y)]$

The image α of c is called a geodesic (or metric) segment joining x and y. We say X is (i) a geodesic space if any two points of X are joined by a geodesic and (ii) uniquely geodesic if there is exactly one geodesic joining x and y for each x, y ∈ X, which we will denote by [x, y], called the segment joining x to y.
Comparision triangle
A geodesic triangle $\Delta(p, q, r)$. in a geodesic metric space (X, d) consists of three points in $p, q, r \in X$ and a geodesic segment between each pair of vertices $[p, q], [q, r], [r, p]$.
A comparison triangle for the geodesic triangle $\Delta(p, q, r)$ in (X, d) is a triangle $\bar{\Delta}\ (\bar{p}, \bar{q}, \bar{r}) \subset \mathbb{R}^2$ such that:

- $d(p, q) = d\ (\bar{p}, \bar{q})$
- $d(q, r) = d\ (\bar{q}, \bar{r})$
- $d(r, p) = d\ (\bar{r}, \bar{p})$

Definition of CAT(0) space

Let (X, d) be a geodesic metric space. It is called CAT(0) space if for any geodesic triangle $\Delta \in X$ and $x, y \in \Delta$:

$d(x, y) \leq \mathrm{d}\,(\bar{x}, \bar{y})$ ku $\bar{x}, \bar{y} \in \bar{\Delta}$

Main Result

Generalization of T-Ciric Quasi Contraction Mapping

Let X be a CAT(0) space and $S, T : X \to X$ be two mappings. Then S is called T-Ciric quasi contraction mapping if it satisfies the following condition:

(1.1)

$$d(TSx,TSy) \leq h \max\left\{d(Tx,Ty), \frac{d(Tx,TSx)+d(Ty,TSy)}{2}, \frac{d(Tx,TSy)+d(Ty,TSx)}{2}\right\}$$

$(TCQC)$

for all $x, y \in X$ and $0 < h < 1.$

Then the condition (TCQC) can be generalized as follows:

(4.18)

$$d(TSx,TSy) \leq h \max\left\{d(Tx,Ty), \frac{d(Tx,TSx)+d(Ty,TSy)}{m}, \frac{d(Tx,TSy)+d(Ty,TSx)}{m}\right\}$$

$(TCQC)^*$

for all $x, y \in X$ and $0 < h < \frac{m}{2}.$

Proof

Each of the conditions $(TZ_1)-(TZ_3)$ implies $(TCQC)^*$

(TZ_1) $d(TSx,TSy) \leq ad(Tx,Ty) \leq a\frac{m}{2}d(Tx,Ty),\quad 0 < a < 1,\ m \geq 2.$

(TZ_2) $d(TSx,TSy) \leq b\left[d(Tx,TSx)+d(Ty,TSy)\right],\quad 0 < b < \frac{1}{2}$

(TZ_3) $d(TSx,TSy) \leq c\left[d(Tx,TSy)+d(Ty,TSx)\right],\quad 0 < c < \frac{1}{2}$

implies:

$$d(TSx,TSy) \leq \max\left\{a\frac{m}{2}d(Tx,Ty), bm\frac{d(Tx,TSx)+d(Ty,TSy)}{m}, cm\frac{d(Tx,TSy)+d(Ty,TSx)}{m}\right\}$$

$$\leq h\max\left\{d(Tx,Ty),\frac{d(Tx,TSx)+d(Ty,TSy)}{m},\frac{d(Tx,TSy)+d(Ty,TSx)}{m}\right\}$$

when $h=\max\left\{a\frac{m}{2},bm,cm\right\}.$

$$0<a<1\Rightarrow 0<a\frac{m}{2}<\frac{m}{2}$$

$$0<b<\frac{1}{2}\Rightarrow 0<bm<\frac{m}{2} \qquad \Rightarrow 0<h<\frac{m}{2}.$$

$$0<c<\frac{1}{2}\Rightarrow 0<cm<\frac{m}{2}$$

Generalization of strong convergence theorems in CAT(0) spaces Theorem

Let C be a nonempty closed convex subset of a complete CAT(0) space. Let $S,T:C\to C$ be two commuting mappings such that T is continuous, one-to-one, sub-sequentially convergent and $S:C\to C$ is a T-Ciric quasi-contractive operator satisfying (TCQC)* with $0<h<\frac{m}{2}, m\geq 2.$ Let $\{x_n\}$ be defined by the iteration scheme (1.8) [1] . If $\sum_{n=1}^{\infty}\alpha_n=\infty,\ \sum_{n=1}^{\infty}\alpha_n\beta_n=\infty,\ \sum_{n=1}^{\infty}\alpha_n\beta_n\gamma_n=\infty,$ then $\{Tx_n\}$ converges strongly to Tu, where u is the fixed point of the operator S in C.

Proof

From Theorem 1.1 [1], we get that *S* has a unique fixed point in *C*, say *u*. Consider $x,y\in C$. Since S in a T-Ciric quasi-contractive operator satisfying (TCQC)*, then if

$$d(TSx,TSy)\leq\frac{h}{m}\left[d(Tx,TSx)+d(Ty,TSy)\right]$$

$$\leq\frac{h}{m}\left[d(Tx,TSx)+d(Ty,Tx)+d(Tx,TSx)+d(TSx,TSy)\right],$$

Implies

$$\left(1-\frac{h}{m}\right)d(TSx,TSy)\leq\frac{h}{m}d(Tx,Ty)+\frac{2h}{m}d(Tx,TSx),$$

Which yields (using the fact that $0<h<\frac{m}{2},m\geq 2)$

$$d(TSx,TSy) \le \left(\frac{h/m}{1-h/m}\right) d(Tx,Ty) + \left(\frac{2h/m}{1-h/m}\right) d(Tx,TSx).$$

If

$$d(TSx,TSy) \le \frac{h}{m}\left[d(Tx,TSy)+d(Ty,TSx)\right]$$

$$\le \frac{h}{m}\left[d(Tx,TSx)+d(TSx,TSy)+d(Ty,Tx)+d(Tx,TSx)\right]$$

Implies

$$\left(1-\frac{h}{m}\right) d(TSx,TSy) \le \frac{h}{m} d(Tx,Ty) + \frac{2h}{m} d(Tx,TSx)$$

Which also yields (using the fact that $0 < h < \frac{h}{m}, m \ge 2)$

(4.9) $$d(TSx,TSy) \le \left(\frac{h/m}{1-h/m}\right) d(Tx,Ty) + \left(\frac{2h/m}{1-h/m}\right) d(Tx,TSx).$$

Denote

$$\delta = \max\left\{h, \frac{h/m}{1-h/m}\right\} = h,$$

$$L = \frac{2h/m}{1-h/m}.$$

Thus,in all cases,

$$d(TSx,TSy) \le \delta d(Tx,Ty) + Ld(Tx,TSx)$$

(4.20) $$= hd(Tx,Ty) + \left(\frac{2h/m}{1-h/m}\right) d(Tx,TSx).$$

holds for all $x, y \in C$.

Also from (TCQC)* with $y = u = Su$, we have

$$d(TSx,TSu) \le h\max\left\{d(Tx,Tu), \frac{d(Tx,TSx)}{m}, \frac{d(Tx,TSu)+d(Tu,TSx)}{m}\right\}$$

$$\le h\max\left\{d(Tx,Tu), \frac{d(Tx,Tu)+d(Tu,TSx)}{m}, \frac{d(Tx,TSu)+d(Tu,TSx)}{m}\right\}$$

$$= h\max\left\{d(Tx,Tu), \frac{d(Tx,Tu)+d(Tu,TSx)}{m}\right\}$$

(4.21) $$\le hd(Tx,Tu).$$

Now (4.21) gives

(4.22) $$d(TSx_n, Tu) \le hd(Tx_n, Tu).$$

(4.23) $$d(TSy_n, Tu) \le hd(Ty_n, Tu).$$

(4.24) $$d(TSz_n, Tu) \le hd(Tz_n, Tu).$$

Using (1.8),(2.6) and Lemma 1.1(ii) [1], we have

$$\begin{aligned} d(Tz_n, Tu) &= d(\gamma_n TSx_n \oplus (1-\gamma_n)Tx_n, Tu) \\ &\le \gamma_n d(TSx_n, Tu) + (1-\gamma_n) d(Tx_n, Tu) \\ &\le \gamma_n hd(Tx_n Tu) + (1-\gamma_n) d(Tx_n, Tu) \\ &\le [1-(1-h)\gamma_n] d(Tx_n, Tu). \end{aligned} \tag{4.25}$$

Again using (1.8),(2.5),(2.7) and Lemma 1.1(ii) [1], we have

$$\begin{aligned} d(Ty_n, Tu) &\le d(\beta_n TSz_n \oplus (1-\beta_n)Tx_n, Tu) \\ &\le \beta_n d(TSz_n, u) + (1-\beta_n) d(Tx_n, Tu) \\ &\le \beta_n hd(Tz_n Tu) + (1-\beta_n) d(Tx_n, Tu) \\ &\le \beta_n h[1-(1-h)\gamma_n] d(Tx_n Tu) + (1-\beta_n) d(Tx_n, Tu) \\ &\le [1-(1-h)\beta_n - h(1-h)\beta_n\gamma_n] d(Tx_n, Tu). \end{aligned} \tag{4.26}$$

Now using (1.8),(2.4),(2.8), $TS = ST$ (by assumption of the theorem) and Lemma 1.7(ii) [1], we have

$$d(Tx_{n+1}, Tu) = d(\alpha_n STy_n \oplus (1-\alpha_n)Tx_n, Tu)$$

$$\begin{aligned} &\le \alpha_n d(STy_n, Tu) + (1-\alpha_n) d(Tx_n, Tu) \\ &\le \alpha_n hd(Ty_n, Tu) + (1-\alpha_n) d(Tx_n, Tu) \\ &\le \alpha_n h[1-(1-h)\beta_n - h(1-h)\beta_n\gamma_n] d(Tx_n, Tu) + (1-\alpha_n) d(Tx_n, Tu) \\ &\le [1-\{(1-h)\alpha_n - h(1-h)\beta_n\gamma_n + h^2(1-h)\alpha_n\beta_n\gamma_n\}] d(Tx_n, Tu) \\ &= (1-\beta_n) d(Tx_n, Tu), \end{aligned}$$

Where $\beta_n = \{(1-h)\alpha_n - h(1-h)\alpha_n\beta_n + h^2(1-h)\alpha_n\beta_n\gamma_n\},$ since $0 < h < \frac{m}{2}, m \ge 2,$ and by assumption of the theorem $\sum_{n=1}^{\infty} \alpha_n = \infty,$ $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty,$ $\sum_{n=1}^{\infty} \alpha_n\beta_n\gamma_n = \infty,$ *it follows that* $\sum_{n=1}^{\infty} \beta_n = \infty,$ therefore by Lemma 1.8 [1], we get that $\lim_{n\to\infty} d(Tx_n, Tu) = 0.$ Therefore $\{Tx_n\}$ converges strongly to $Tu,$ where *u* is the fixed point of the operator *S* in *C*. This completes the proof. □

Corollary 1

Let C be a nonempty closed convex subset of a complete CAT(0) space. Let $S, T : C \to C$ *be two commuting mappings such that T is continuous, one-to-one, subsequentially convergent and* $S : C \to C$ *is T-Kannan contractive operator satisfying the condition*

$$d(TSx, TSy) \le b\left[\frac{d(Tx, TSx) + d(Ty, TSy)}{m}\right],$$

$$\forall x, y \in X; b \in \left(0, \frac{1}{m}\right), \forall m \ge 2.$$

Let $\{Tx_n\}$ *be defined by the iteration scheme (1.8) [1]. If* $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\sum_{n=1}^{\infty}\alpha_n\beta_n = \infty$, *and* $\sum_{n=1}^{\infty}\alpha_n\beta_n\gamma_n = \infty$, *then* $\{Tx_n\}$ *converges strongly to Tu, where u is the fixed point of the operator S in C.*

Corollary 2

Let C be a nonempty closed convex subset of a complete CAT(0) space. Let $S, T : C \to C$ *be two commuting mappings such that T is continuous, one-to-one, subsequentially convergent and* $S : C \to C$ *is T-Chatterjea contractive operator satisfying the condition*

$$d(TSx, TSy) \le c\left[\frac{d(Tx, TSx) + d(Ty, TSy)}{m}\right],$$

$$\forall x, y \in X; c \in \left(0, \frac{1}{m}\right), \forall m \ge 2.$$

Let $\{Tx_n\}$ *be defined by the iteration scheme (1.8) [1]. If* $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\sum_{n=1}^{\infty}\alpha_n\beta_n = \infty$, *and* $\sum_{n=1}^{\infty}\alpha_n\beta_n\gamma_n = \infty$, *then* $\{Tx_n\}$ *converges strongly to Tu, where u is the fixed point of the operator S in C.*

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