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Generalization of strong convergence theorem in CAT(0) spaces

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Abstract. The aim of this paper is to give the generalization condition of T-Ciric quasi contractive mapping. Also to study the generalization of strong convergence theorem of modified S-iteration process for Ciric quasi contractive operator in the framework of CAT(0) spaces based on new generalized condition for T-Ciric quasi contractive mapping. Our results extend and generalize many known results from the previous work given in the existing literature (see [1,6]).

1. Introduction and Preliminaries

CAT(0)space. A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces (see [3]), R-trees (see [11]), Euclidean buildings (see [12]), the complex Hilbert ball with a hyperbolic metric (see [13]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [3]. Fixed point theory in CAT(0) spaces was first studied by Kirk (see [1,2]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then, the fixed point theory for single-valued and multi-valued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared.

Let (X, d) be a metric space. A geodesic path joining $x, y \in X$ is a map $c : [0, d(x, y)] \rightarrow X$ such that:

- c(0) = x
- c(d(x,y)) = y
- $d(c(t_1), c(t_2)) = |t_1 t_2|$, $\forall t_1, t_2 \in [0, d(x, y)]$

The image α of c is called a geodesic (or metric) segment joining x and y. We say X is (i) a geodesic space if any two points of X are joined by a geodesic and (ii) uniquely geodesic if there is exactly one geodesic joining x and y for each x, $y \in X$, which we will denote by [x, y], called the segment joining x to y.

Comparision triangle

A geodesic triangle $\Delta(p, q, r)$. in a geodesic metric space (X, d) consists of three points in $p, q, r \in X$ and a geodesic segment between each pair of vertices [p, q], [q, r], [r, p].

A comparison triangle for the geodesic triangle $\Delta(p, q, r)$ in (X, d) is a triangle $\overline{\Delta}(\overline{p}, \overline{q}, \overline{r}) \subset \mathbb{R}^2$ such that:

- $d(p,q) = d(\bar{p},\bar{q})$
- $d(q,r) = d(\bar{q},\bar{r})$
- $d(r,p) = d(\bar{r},\bar{p})$

Generalization of strong convergence theorem in CAT(0) spaces

Definition of CAT(0) space

Let (X, d) be a geodesic metric space. It is called CAT(0) space if for any geodesic triangle $\Delta \in X$ and $x, y \in \Delta$: $d(x, y) \leq d(\bar{x}, \bar{y})$ ku $\bar{x}, \bar{y} \in \bar{\Delta}$

Main Result

Generalization of T-Ciric Quasi Contraction Mapping

Let X be a CAT(0) space and $S, T : X \to X$ be two mappings. Then S is called T-Ciric quasi contraction mapping if it satisfies the following condition: (1.1)

$$d(TSx, TSy) \le h \max\left\{ d(Tx, Ty), \frac{d(Tx, TSx) + d(Ty, TSy)}{2}, \frac{d(Tx, TSy) + d(Ty, TSx)}{2} \right\}$$

$$(TCOC)$$

for all $x, y \in X$ and 0 < h < 1.

Then the condition (TCQC) can be generalized as follows: (4.18)

$$d(TSx, TSy) \le h \max\left\{ d(Tx, Ty), \frac{d(Tx, TSx) + d(Ty, TSy)}{m}, \frac{d(Tx, TSy) + d(Ty, TSx)}{m} \right\}$$
$$(TCQC)^*$$

for all $x, y \in X$ and $0 < h < \frac{m}{2}$.

Proof

Each of the conditions $(TZ_1) - (TZ_3)$ implies $(TCQC)^*$

$$(IZ_{1}) \quad d(TSx, TSy) \leq ad(Tx, Ty) \leq a\frac{m}{2}d(Tx, Ty), \quad 0 < a < 1, \ m \geq 2.$$

$$(TZ_{2}) \quad d(TSx, TSy) \leq b[d(Tx, TSx) + d(Ty, TSy)], \quad 0 < b < \frac{1}{2}$$

$$(TZ_{3}) \quad d(TSx, TSy) \leq c[d(Tx, TSy) + d(Ty, TSx)], \quad 0 < c < \frac{1}{2}$$

implies:

$$d(TSx, TSy) \le \max\left\{a\frac{m}{2}d(Tx, Ty), bm\frac{d(Tx, TSx) + d(Ty, TSy)}{m}, cm\frac{d(Tx, TSy) + d(Ty, TSx)}{m}\right\}$$

$$\leq h \max\left\{d(Tx,Ty), \frac{d(Tx,TSx) + d(Ty,TSy)}{m}, \frac{d(Tx,TSy) + d(Ty,TSx)}{m}\right\}$$

when $h = \max\left\{a\frac{m}{2}, bm, cm\right\}$.
 $0 < a < 1 \Rightarrow 0 < a\frac{m}{2} < \frac{m}{2}$
 $0 < b < \frac{1}{2} \Rightarrow 0 < bm < \frac{m}{2} \Rightarrow 0 < h < \frac{m}{2}$.
 $0 < c < \frac{1}{2} \Rightarrow 0 < cm < \frac{m}{2}$

Generalization of strong convergence theorems in CAT(0) spaces Theorem

Let C be a nonempty closed convex subset of a complete CAT(0) space. Let $S, T : C \to C$ be two commuting mappings such that T is continuous, one-to-one, sub-sequentially convergent and $S: C \to C$ is a T-Ciric quasi-contractive operator satisfying (TCQC)^{*} with $0 < h < \frac{m}{2}, m \ge 2$. Let $\{X_n\}$ be defined by the iteration scheme (1.8) [1]. If $\sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} \alpha_n \beta_n = \infty, \sum_{n=1}^{\infty} \alpha_n \beta_n \gamma_n = \infty$, then $\{Tx_n\}$ converges strongly to Tu, where u is the fixed point of the operator S in C.

Proof

From Theorem 1.1 [1], we get that *S* has a unique fixed point in *C*, say *u*. Consider $x, y \in C$. Since *S* in a T-Ciric quasi-contractive operator satisfying (TCQC)^{*}, then if

$$d(TSx, TSy) \leq \frac{h}{m} \Big[d(Tx, TSx) + d(Ty, TSy) \Big]$$

$$\leq \frac{h}{m} \Big[d(Tx, TSx) + d(Ty, Tx) + d(Tx, TSx) + d(TSx, TSy) \Big],$$

Implies

$$\left(1 - \frac{h}{m}\right) d(TSx, TSy) \le \frac{h}{m} d(Tx, Ty) + \frac{2h}{m} d(Tx, TSx),$$

$$0 < h < \frac{m}{2}, m \ge 2$$

Which yields (using the fact that

$$d(TSx, TSy) \leq \left(\frac{h/m}{1-h/m}\right) d(Tx, Ty) + \left(\frac{2h/m}{1-h/m}\right) d(Tx, TSx).$$

If

$$d(TSx, TSy) \leq \frac{h}{m} [d(Tx, TSy) + d(Ty, TSx)]$$
$$\leq \frac{h}{m} [d(Tx, TSx) + d(TSx, TSy) + d(Ty, Tx) + d(Tx, TSx)]$$

Implies

$$(1 - \frac{h}{m}) d(TSx, TSy) \le \frac{h}{m} d(Tx, Ty) + \frac{2h}{m} d(Tx, TSx)$$

$$0 < h < \frac{h}{m}, m \ge 2$$
where the fact that that tha

(4.9)
$$d(TSx, TSy) \le \left(\frac{h/m}{1-h/m}\right) d(Tx, Ty) + \left(\frac{2h/m}{1-h/m}\right) d(Tx, TSx).$$

Denote

(4.20)

$$\delta = \max\left\{h, \frac{h/m}{1 - h/m}\right\} = h,$$
$$L = \frac{2h/m}{1 - h/m}.$$

Thus, in all cases,

$$d(TSx, TSy) \le \delta d(Tx, Ty) + Ld(Tx, TSx)$$

$$= hd(Tx,Ty) + \left(\frac{2h/m}{1-h/m}\right)d(Tx,TSx).$$

holds for all $x, y \in C$.

Also from $(TCQC)^*$ with y = u = Su, we have

$$d(TSx, TSu) \le h \max\left\{d(Tx, Tu), \frac{d(Tx, TSx)}{m}, \frac{d(Tx, TSu) + d(Tu, TSx)}{m}\right\}$$
$$\le h \max\left\{d(Tx, Tu), \frac{d(Tx, Tu) + d(Tu, TSx)}{m}, \frac{d(Tx, TSu) + d(Tu, TSx)}{m}\right\}$$
$$= h \max\left\{d(Tx, Tu), \frac{d(Tx, Tu) + d(Tu, TSx)}{m}\right\}$$
$$(4.21) \le hd(Tx, Tu).$$

Now (4.21) gives

 $(4.22) d(TSx_n, Tu) \le hd(Tx_n, Tu).$

$$(4.23) d(TSy_n, Tu) \le hd(Ty_n, Tu).$$

$$(4.24) d(TSz_n, Tu) \le hd(Tz_n, Tu).$$

Using (1.8),(2.6) and Lemma 1.1(ii) [1], we have

$$d(Tz_n, Tu) = d(\gamma_n TSx_n \oplus (1 - \gamma_n)Tx_n, Tu)$$

$$\leq \gamma_n d(TSx_n, Tu) + (1 - \gamma_n)d(Tx_n, Tu)$$

$$\leq \gamma_n hd(Tx_nTu) + (1 - \gamma_n)d(Tx_n, Tu)$$
(4.25)

 $\leq [1-(1-h)\gamma_n]d(Tx_n,Tu).$

Again using (1.8),(2.5),(2.7) and Lemma 1.1(ii) [1], we have $d(T_{Y}, T_{Y}) \leq d(R T S_{Z} \oplus (1 - R) T_{Y}, T_{Y})$

(4.26)
$$a(Ty_{n}, Tu) \leq a(\rho_{n}TSz_{n} \oplus (1-\rho_{n})Tx_{n}, Tu) \leq \beta_{n}d(TSz_{n}, u) + (1-\beta_{n})d(Tx_{n}, Tu) \leq \beta_{n}hd(Tz_{n}Tu) + (1-\beta_{n})d(Tx_{n}, Tu) \leq \beta_{n}h[1-(1-h)\gamma_{n}]d(Tx_{n}Tu) + (1-\beta_{n})d(Tx_{n}, Tu) \leq [1-(1-h)\beta_{n}-h(1-h)\beta_{n}\gamma_{n}]d(Tx_{n}, Tu).$$

Now using (1.8),(2.4),(2.8), TS = ST (by assumption of the theorem) and Lemma 1.7(ii) [1], we have

$$d(Tx_{n+1}, Tu) = d(\alpha_n STy_n \oplus (1 - \alpha_n)Tx_n, Tu)$$

$$\leq \alpha_n d(STy_n, Tu) + (1 - \alpha_n)d(Tx_n, Tu)$$

$$\leq \alpha_n hd(Ty_n, Tu) + (1 - \alpha_n)d(Tx_n, Tu)$$

$$\leq \alpha_n h[1 - (1 - h)\beta_n - h(1 - h)\beta_n\gamma_n]d(Tx_n, Tu) + (1 - \alpha_n)d(Tx_n, Tu)$$

$$\leq [1 - \{(1 - h)\alpha_n - h(1 - h)\beta_n\gamma_n + h^2(1 - h)\alpha_n\beta_n\gamma_n\}]d(Tx_n, Tu)$$

$$= (1 - \beta_n)d(Tx_n, Tu),$$
Where $\beta_n = \{(1 - h)\alpha_n - h(1 - h)\alpha_n\beta_n + h^2(1 - h)\alpha_n\beta_n\gamma_n\},$ since $0 < h < \frac{m}{2}, m \ge 2,$ and by assumption of the theorem $\sum_{n=1}^{\infty} \alpha_n = \infty,$

$$\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty, \sum_{n=1}^{\infty} \alpha_n \beta_n \gamma_n = \infty, it \text{ follows that } \sum_{n=1}^{\infty} \beta_n = \infty, \text{ therefore by Lemma 1.8 [1], we get that $\lim_{n \to \infty} d(Tx_n, Tu) = 0.$ Therefore $\{Tx_n\}$ converges strongly to Tu , where u is the fixed point of the operator S in C . This completes the proof. $\Box$$$

Corollary 1

Let C be a nonempty closed convex subset of a complete CAT(0) space. Let $S,T: C \to C$ be two commuting mappings such that T is continuous, one-to-one, subsequentially convergent and $S: C \to C$ is T-Kannan contractive operator satisfying the condition

$$d(TSx, TSy) \le b \left[\frac{d(Tx, TSx) + d(Ty, TSy)}{m} \right]$$
$$\forall x, y \in X; b \in \left(0, \frac{1}{m}\right), \forall m \ge 2.$$

Let $\{Tx_n\}$ be defined by the iteration scheme (1.8) [1]. If $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ then $\{Tx_n\}$ converges strongly to Tu where u

 $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty, \text{ and } \sum_{n=1}^{\infty} \alpha_n \beta_n \gamma_n = \infty, \text{ then } \{Tx_n\} \text{ converges strongly to Tu, where u is the fixed point of the operator S in C.}$

Corollary 2

Let C be a nonempty closed convex subset of a complete CAT(0) space. Let $S, T : C \to C$ be two commuting mappings such that T is continuous, one-to-one, subsequentially convergent and $S: C \to C$ is T-Chatterjea contractive operator satisfying the condition

$$d(TSx, TSy) \le c \left[\frac{d(Tx, TSx) + d(Ty, TSy)}{m} \right],$$

$$\forall x, y \in X; c \in \left(0, \frac{1}{m}\right), \forall m \ge 2.$$

Let $\{Tx_n\}$ be defined by the iteration scheme (1.8) [1]. If $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$, and $\sum_{n=1}^{\infty} \alpha_n \beta_n \gamma_n = \infty$, then $\{Tx_n\}$ converges strongly to Tu, where u is the fixed point of the operator S in C.

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