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Constructing Hadamard matrices using binary codes

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Abstract: In this paper is presented a very efficient method for constructing Hadamard matrices, using binary code products. We will construct such matrices using the scalar production of two vectors and the tensor production of Hadamard matrices. This method is based on the representation of the natural number as a binary code which takes only two values 0 or 1. Such a method of generating Hadamard matrices can be used in practice to generate different codes, in telecommunication systems, to correct blocked codes, but also in science as for example in Boolean algebra.

Keywords: Hadamad matrices, binary code, scalar product, tensorial product.

1 INTRADUCTION

Definition of Hadamard matrix:[8] A Hadamard matrix of order n, H_n, is $n \times n$, a square matrix with elements $+1$'s and -1's such that $H_n \cdot H_n^T = nI_n$, where I_n is the identity matrix of order n.

Definition 2. [3] The Hadamard matrix which has all the elements (components) in the first row and column 1 is called the normalized Hadamard matrix.

Examples of the normalized Hadamard matrix of order 2 and 4:

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

If $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ are vector over ����, then, scalar product defined: ���� = (��������,

��������, ... ��������). [9]

The Kronecker product also called tensor product or the direct product of two matrices �� ������ �� is defined as follows:

�� ⊗ �� = (�������� ������������� ��������⋯ ������������� ����⋯ ����)

���� = (�� �� ⋮ ⋮ ⋱ ⋮ �������� �������� ⋯ ��������

For example,

���� = ���� ⊗ �� �� �� −��|�� �� �� −�� �� �� �� ��

�� −��) ⊕ (�� �� �� �� �� −��

�� �� −��) = (−��|−�� −�� −�� .) = (�� �� −�� −�� �� −�� −�� ��)

�� �� � −��

3 CONSTRUCTION OF THE BINARY CODES

For the construction of Hadamard matrices using binary codes, we will give a field, based on which we can define the elements of the Hadamard matrix.

Lemma 1.[5] Let $u_0 i_1 i_2 \ldots i$ $j= k- k-$, when ����,���� $i= k- k-$ and $u_0 j_1 j_2 \ldots j$ ������(����,���� − ���������������������� ���)

then:

$\Sigma \quad ( ) ( )^{k-1}$

$H_k i j$ j_r $r i$

when $_2$, $1 = -$ $r = 0$

1

$k -$

$\sum =$

$( )$ $r r i j$

, $r = 0$ is scalar production. $i j u , u$

Proof: We will do the proof by mathematical induction, according ��.

For �� = �� confirmation it is trivial:

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Assume that the lemma is valid for ����, ku �� = ����and prove that:

$$H_2 = \begin{bmatrix} H & H \\ H & -H \end{bmatrix} \quad n \; n \quad n \quad n \; n$$

The four possible cases for (��,��) are: $(0,0),(0,1),(1,0),(1,1)$.

For (��,��) = (��, ��) ���� (��, ��) ���� (��, ��) we have:

$( ) ( ) ( ) ( )$ $2\,2\,2\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0$ $H\, i, j\; H\, u ,u\; H\, i\, i \ldots i , j\, j \ldots j\; H\, i \ldots i , j \ldots j$ $n = n\; i$

$j = n\; k\; k- k\; k- = n\; k- k-$

For, by mathematical induction we get the equation:

$1 \quad k$

$k -$

$- \; 0 - \sum$

$$\sum$$

$$(1)\,(1)\ {}^{0}{}_{i\,j\,i\,j}$$

$$r\,r \qquad r\,r$$

$$=$$

[1]

$$r_{=} \qquad r_{=}$$

For (��,��) = (��, ��) we have:

$$(\ )\,(\ )\,(\ )\ 2\,2\,1\,0\,1\,0\,1\,0\,1\,0\ H\,i,j\ H\,i\,i\ \ldots i\ ,j\,j\ \ldots j\ H\,i\ \ldots i\ ,j\ \ldots j\ {}_{n}={}_{n}$$

$$k\,k-\,k\,k- = -\ n\,k-\,k-$$

Using the inductive assumption we will have:

$$- \ 1 \quad k \quad k \quad k$$

$$2,\ -\ -\ -\sum$$

$$\sum \qquad \sum$$

$$(\ )\,(\ )(\ )\,1\ (1)\ {}_{0}(1)\ {}_{0}$$

$$i\,j\,i\,j\,i\,j$$

$$_{n}H\,i\,j \qquad {}^{+}\,1,1$$

$$= -\ {}_{==}$$

$$r_{=} \quad {}^{0}{}_{r\,r} \quad = \quad {}^{r}{}_{r\,r} \quad = \quad {}^{r}{}_{r\,r}$$

The construction through lemma1 is realized as follows:

We get all binary ��-sets$u\ u\ u\ {}^{k\,1\,2}{}_{2}$

, ,...,. The order of the binary -s will be denoted by that taken from the binary -s, adding 0 first as the component after and then 1, which are elements of

$[0,1]\ \pi_1 =$. So if,$[\ ]^{i}$

$_{i}\,u_1\,u_2\,u_2\,\pi =$, ,...,then:

$$_{+1} = \left[u_1\,0,u_2\,0,\ldots,\ u_2\,0,u_1\,1,u_2\,1,\ldots,u_2\,1\right],\ \forall i \in \{1,2,\ldots,\ k-1\}$$

$$i\ i$$

$$\pi_{\ i}$$

For exmple,

��₍�� = [��, ��], ���� = [����, ����, ����, ����], ���� =
[������, ������, ������, ������, ������, ������,
������, ������]

���� =
[��������, ��������, ��������, ��������, ��������,
��������, ��������, ��������, ��������, ��������,
��������, ��������, ��������, ��������, ��������,
��������]

Let ^{k}

n = 2, and let �� �� -sets under the proper binary ranking as follows:

[ ] 0 1 2 1

, , ,..., π $_k$ = u u u u^{n-}. [5]

We will use this indexing to find the elements, and that is why we start the indexing of 0. For example for binary sets we will have:

with elements ( )$^{( )}$

Dec	Hex	Bin
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000

Matrix,

9	9	1001
10	A	1010
11	B	1011
12	C	1100
13	D	1101
14	E	1110
15	F	1111

$H = (h_{ij})_{(n-1)\times(n-1)}$, $h_{ij} = (-1)^{(u_i,u_j)}$, $\forall i,j \in \{0,1,\ldots,n-1\}$, where $u_i, u_j \in \pi_k$, is a H-matrix of order $n = 2^k$. As a concrete example we will examine $k = 4$ meaning that we will construct the H matrix of order 16, then H_{16}. [5]

Since $u_0 \leftrightarrow 0000$, then it follows that $(u_0,u_i) = (u_j,u_0) = 0$, $\forall i,j \in \{1,2,\ldots,n-1\}$ comes from $h_{0i} = h_{j0} = +1$. h_{ii} will also be +1 or -1 depending on the weight $\omega(u_i)$, where u_i is an even or odd number (number of members 1).

So,

$$h_{3,3} = h_{5,5} = h_{6,6} = h_{9,9} = h_{10,10} = h_{12,12} = h_{15,15} = +1,$$

whereas,

$$h_{1,1} = h_{2,2} = h_{4,4} = h_{7,7} = h_{8,8} = h_{11,11} = h_{13,13} = h_{14,14} = -1.$$

Other elements of the matrix are similarly assigned, such as:

����,�� = +��, because (����,����) = (��������, ��������) = �� · �� + �� · �� + �� · �� + �� · �� = �� and (−��)�� = ��.

������,�� = −��, because (������, ����) = (��������, ��������) = �� · �� + �� · �� + �� · �� + �� · �� = �� and (−��)�� = ��, etc. [1]

So the H-matrix constructed in this way will look like this:

�� �� �� �� �� �� �� ��

�� �� �� �� �� �� �� ��

Now for practical reasons, in the above matrix, we will substitute, 1-s with + and 0-s with -, from where we get the matrix:

```
++++++++++++++++
+--+-++--++-+--+
+-+-+-+-+-+-+-+-
++--++--++--++--
+--++--++--++--+
++++----++++----
+-+--+-++-+--+-+
++----++++----++
++--++--+-++--++-++++
---------+++++-+---+-+-
```

$= {}_{4} = {}^{16}{}_{2}H\,H$

```
+--+-++-+--+-++-+++++-++-+-++------++--++++
+++-----------+-+-+-+---+--
-+-+-++++--++-----+++---+
```

A more complete explanation of the construction of Hadamard matrices with the binary code method will be given below.

Step 1: [2]We obtain the Hadamard matrix of order N as follows:

$$H_N = \begin{bmatrix} h_{00} & h_{01} & h_{02} & \cdots & h_{0m} \\ h_{10} & h_{11} & h_{12} & \cdots & h_{1m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{k0} & h_{k1} & h_{k2} & \cdots & h_{km} \end{bmatrix}$$

where h_{km} are the elements of the Hadamard matrix $\forall k,m = 0,1,2,..., N-1$. Its elements can be found as in the following steps.

Step 2: [1]For $k,m \geq 0$ write the binary numbers k,m as follows:

$$(k)_2 = (k_{n-1} k_{n-2} \ldots k_1 k_0)_b = \sum_{i=0}^{n-1} k_i 2^i$$

$$(m)_2 = (m_{n-1} m_{n-2} \ldots m_1 m_0)_b = \sum_{i=0}^{n-1} m_i 2^i,$$

where,

$$k_i \in \{\ \}\ \forall i = \ldots, n-1$$

1,1 , 0,1,2,..., 1 i

{ 1,1}, 0,1,2,..., 1

m i n

i

where �� = $\log_2$ N.

n − 1

∈ − ∀ = −

Take , (1)

∑

Step 3: [1]

− 0

k m

$h_{k\,m}$, where ⊕represents summation according to module 2,

=

i = ⊕ i i

$$(i - e - 0 \oplus 0 = 0,\ 0 \oplus 1 = 1 \oplus 0 = 1\ dhe\ 1 \oplus 1 = 1)$$

Example. We know that the Hadamard matrix of order 2

is: 1 1

H_2 = 1 1

The Hadamard matrix of the order $N = 2$is taken from the matrix:

h h

H (1)

= 00 01

${}^2h$ h

10 11

All elements of the Hadamard matrix of the order ( ) 0,0 0,1 1,0 1,1 $N = 2$, h ,h ,h ,$h$can be found using the following binary codes:

First, we get $k,m = 0,1$binary numbers: ( ) ( )2 0 = 0 $b$and ( ) ( ) 1 b = 1 2. Then we apply the formula

$$h_{k,m}[1] = \sum_{i=0}^{n-1} (-1)^{k_i m_i \oplus 0}$$

as follows:

$$h_{00} = h^{*}_{0,0} = \sum_{i=0}^{0} (-1)^{k_i m_i \oplus 0} = (-1)^{0 \cdot 0 \oplus 0} = (-1)^{0} = 1$$

$$h_{01} = h^{*}_{0,1} = \sum_{i=0}^{0} (-1)^{k_i m_i \oplus 0} = (-1)^{0 \cdot 1 \oplus 0} = (-1)^{0} = 1$$

$$h_{10} = h^{*}_{1,0} = \sum_{i=0}^{0} (-1)^{k_i m_i \oplus 0} = (-1)^{1 \cdot 0 \oplus 0} = (-1)^{0} = 1$$

$$h_{11} = h^{*}_{1,1} = \sum_{i=0}^{0} (-1)^{k_i m_i \oplus 0} = (-1)^{1 \cdot 1 \oplus 0} = (-1)^{1} = -1$$

Substitute the above equations into equation (1) and obtain:

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$= 1 \; 1$

The Hadamard order $N = 4$ matrix can be constructed using binary codes. The Hadamrd matrix of the order $N = 4$ is given in the form:

$$H_4^{(2)} = \begin{bmatrix} h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & h_{33} \end{bmatrix}$$

The Hadamard order $N = 2^2 = 4$ matrix can be constructed as follows:

First we get binary codes k,m:

$$(0)_2 = (0,0)_b,\; (1)_2 = (0,1)_b,\; (2)_2 = (1,0)_b,\; (3)_2 = (1,1)_b$$

then, we assign the elements of the matrix H_4.[1]

To determine the elements of matrix H_4 we use the formula:

$$h_{k,m} = (-1)^{\sum_{i=0}^{n-1} k_i \oplus m_i}$$

We have:

$$h_{00} = h_{0,0\,0,0} = (-1)^{\sum_{i=0}^{1} k_i \oplus m_i} = (-1)^{(0 \oplus 0)+(0 \oplus 0)} = (-1)^{0+0} = (-1)^{0} = 1$$

$$h_{01} = h_{0,0\,0,1} = (-1)^{\sum_{i=0}^{1} k_i \oplus m_i} = (-1)^{(0 \oplus 0)+(0 \oplus 1)} = (-1)^{0+1} = (-1)^{1} = -1$$

1

= = − ∑ = − = − = − = ( )( ) ( ) ( )^( ) ( )( ) ( )

h h *k m*
0,1 0,0 0 0 0 ⊕ ⊕

1 1 1 1 1

0

02 0,0 1,0 *i i* *i* = ⊕

1

= = − ∑ = − = − = − = ( )( ) ( ) ( )^( ) ( )( ) ( )

h h *k m*
0 1 0 1 0 0 0 * ⊕ * ⊕

1 1 1 1 1

0

03 0,0 1,1 *i i* *i* = ⊕

1

= = − ∑ = − = − = − = ( )( ) ( ) ( )^( ) ( )( ) ( )

h h *k m*
0 0 1 0 0 0 0 * ⊕ * ⊕

1 1 1 1 1

0

10 0,1 0,0 *i i* *i* = ⊕

1

= = − ∑ = − = − = − = − ( )( ) ( ) ( )^( ) ( )( ) ( )

h h *k m*
0 0 1 1 0 1 1 * ⊕ * ⊕

1 1 1 1 1

0

11 0,1 0,1 *i i* *i* = ⊕

1

= = − ∑ = − = − = − = ( )( ) ( ) ( )^( ) ( )

1 1 (1) (1) 1

h h 12 0,1 1,0 *k m* *i i* *i* = ⊕ 0 ⊕
0 1 1 0 0 0 0 * ⊕ *

1

= = − ∑ = − = − = − = − ( )( ) ( ) ( )^( ) ( )( ) ( )

h h *k m*
0 1 1 1 0 1 1 * ⊕ * ⊕

1 1 1 1 1

13 0,1 1,1 *i i* *i* = ⊕ 0

1

= = − ∑ = − = − = − = ( )( ) ( ) ( )^( ) ( )( ) ( )

h h

k m
1 0 0 0 0 0 0 * ⊕ * ⊕

1 1 1 1 1

20 1.0 0,0 i i i = ⊕ 0

1

= = − ∑ = − = − = − = ( )( ) ( ) ( )( ) ( ) ( ) ( )

h h

k m
1 0 0 1 0 0 0 * ⊕ * ⊕

1 1 1 1 1

21 1,0 0,1 i i i = ⊕ 0

1

= = − ∑ = − = − = − = − ( )( ) ( ) ( )( ) ( ) ( ) ( )

h h

k m
1 1 0 0 1 0 1 * ⊕ * ⊕

1 1 1 1 1

22 1,0 1,0 i i i = ⊕ 0

1

= = − ∑ = − = − = − = − ( )( ) ( ) ( )( ) ( ) ( ) ( )

h h

k m
1 1 0 1 1 0 1 * ⊕ * ⊕

1 1 1 1 1

23 1,0 1,1 i i i = ⊕ 0

1

= = − ∑ = − = − = − = ( )( ) ( ) ( )( ) ( )

1 1 (1) (1) 1

h h k m i i i = ⊕ 0 ⊕
30 1,1 0,0
1 0 1 0 0 0 0 * ⊕ *

1

= = − ∑ = − = − = − = − ( )( ) ( ) ( )( ) ( ) ( ) ( )

h h

k m
1 0 1 1 0 1 1 * ⊕ * ⊕

1 1 1 1 1

31 1,1 0,1 i i i = ⊕ 0

1

= = − ∑ = − = − = − = − ( )( ) ( ) ( )( ) ( ) ( ) ( )

h h

k m
1 1 1 0 1 0 1 * ⊕ * ⊕

1 1 1 1 1

32 1,1 1,0 i i i = ⊕ 0

1

= = − ∑ = − = − = − = ( )( )( ) ( )( ) ( )

1 1 (1) (1) 1

h h k m ⊕ ⊕

33 1,1 1,1 i i 0

i 1 1 1 1 1 1 0 * ⊕ *

=

We replace the obtained elements in the matrix (2) and we have:

1 1

1 1

H_4 ||||[[−−

1 1

−

1 1 |||]]

== − −

1 1 1

1 1 1

− 1

1

$^3 N$ = =is taken from the matrix:

Example1. [8] The Hadamard matrix of order 2 8

h h h h h h h h

||||||||||[[

||||||||||]]

00 01 02 03 04 05 06 07 h h h h h h h h 10 11 12 13 14

15 16 17

50 51 52 53 54 55 56 57

h h h h h h h h

60 61 62 63 64 65 66 67

H h h h h h h h h

8 70 71 72 73 74 75 76 77

=

h h h h h h h h

20 21 22 23 24 25 26 27

h h h h h h h h

30 31 32 33 34 35 36 37

h h h h h h h h

40 41 42 43 44 45 46 47

h h h h h h h h $^3 N$ = =order

Acting as in the matrix of order 4, we find all the elements of the matrix 2 8 and have:

1 1 1 1 1 1 1 1

||||||||||[[− − − − ||||||||||]]

H_8

= ‒ ‒ ‒ ‒

1 1 1 1 1 1 1 1

‒ ‒ ‒ ‒

1 1 1 1 1 1 1 1

‒ ‒ ‒ ‒

1 1 1 1 1 1 1 1

‒ ‒ ‒ ‒

3 Conclusions

1 1 1 1 1 1 1 1

1 1 1 1 1 1 1 1 ‒ ‒ ‒ ‒

‒ ‒ ‒ ‒ 1 1 1 1 1 1 1 1

1 1 1 1 1 1 1 1

· Constructing Hadamard matrices using binary codes is easy, using the appropriate equation for assigning matrix elements.

· Binary code construction is quite simple and fast using the vector scalar product. · For practical reasons the elements of the Hadamard matrix {+ 1, -1} are denoted by +, respectively -.

· The Hadamard matrix that has the first row and column 1's, respectively +, is the Hadamard normalized matrix.

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