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Abstract. Laplacian’s is an operator which plays a very important role in various scientific branches. It is an operator which is defined as the nabla square operator. Its role is very important also in relation to the various analyzes of the operational fields that have to do here with heat transfer. Its role is especially considered in relation to the analysis of potential fields, here related to the phenomenon of heat transfer.

Key words: Nabla Operator, Laplacian Operator, Potential Operational Field

Abstrakti : Laplasiani është nje operator i cili luan nje rol shumë të rendesishem në deget më të ndryshme shkencore. Eshte fjala për një operator i cili definohet si operatori nabla në katror. Roli i tij është shumë i rendesishem edhe në lidhje me analizat më të ndryshme te fushave operacionale qe kanë të bejne ketu me transmetimin e nxehtesisë. Roli i tij sidomos është me konsiderate të madhe në lidhje me analizen e fushave potenciale, ketu nderlidhur me fenomenin e transmetimit te nxehtësisë.

Fjalet kyçe: Operatori nabla, Operatori laplasian, Fusha operacionale potenciale

1. Introduction

The Laplacian operator is a second order differential operator, which is very important in relation to the various analyzes in thermotechnics and thermoenergy, which in principle can be defined according to:

$$divgrad(t) = \nabla^2 t = \Delta t \quad (1),$$

and is defined as the Laplace operator, or abbreviated as laplacian.

At orthogonal arc (curvilinear) coordinates laplacian can be given as a vector:

$$gradt = \nabla t = \sum_{i=1}^{3} \frac{\vec{e}_i}{H_i} \frac{\partial t}{\partial \xi_i} \quad (2),$$

so t- is the temperature of the operational field, ξ_i - the corresponding curvilinear coordinate.

The divergence of any vector can be represented by:

$$div\vec{q} = \frac{1}{J} \sum_{i=1}^{3} \frac{\partial}{\partial \xi_i} \left(\frac{1}{H_i} q_i \right) \quad (3).$$

Thus for the respective component can be written:

$$q_i = \frac{1}{H_i} \frac{\partial t}{\partial \xi_i} \quad (4),$$

respectively:

$$\nabla^2 t = \frac{1}{J} \sum_{i=1}^{3} \frac{\partial}{\partial \xi_i} \left(\frac{1}{H_i^2} \frac{\partial t}{\partial \xi_i} \right) \quad (5).$$

Often for the Cartesian coordinate system is taken:

$$x \equiv \xi_1 \text{ ; } y \equiv \xi_2 \text{ ; } z \equiv \xi_3 \text{ ; } H_1 = H_2 = H_3 = 1 \quad (6).$$

For this case the Laplacian operator is given according to:

$$\nabla^2 t = \Delta t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} \quad (7).$$

In the orthogonal cylindrical system respectively it is valid:

$$x = u(\xi_1, \xi_2)\ ;\ y = v(\xi_1, \xi_2)\ ;\ z = \xi_3 \qquad (8),$$

so that u (ξ_1, ξ_2) and v (ξ_1, ξ_2) are the real and imaginary part of the analytical function (ξ_1 + i ξ_2), which meet the Cauchy-Riemann conditions, and the laplacian in the cylindrical coordinate system is given by:

$$\nabla^2 t = \frac{1}{J}\left(\frac{\partial^2 t}{\partial \xi_1^2} + \frac{\partial^2 t}{\partial \xi_2^2}\right) + \frac{\partial^2 t}{\partial z^2} \qquad (9),$$

where J is Jacobean to the transformation of the corresponding coordinates.

For the planned operational area the previous dependence is strengthened:

$$\nabla^2 t = \frac{1}{J}\left(\frac{\partial^2 t}{\partial \xi_1^2} + \frac{\partial^2 t}{\partial \xi_2^2}\right) \qquad (10).$$

For spherical cylindrical system as a special case can be written:

$$x + iy = \exp(\xi_1 + i\xi_2) \qquad (11),$$

so that $i^2 = -1$ is understood to be the imaginary unit.

Since the Lame parameters are now equal:

$$H_1 = H_2 = \exp \xi_1 \qquad (12),$$

then Jacobean is now:

$$J = \exp(2\xi_1) \qquad (13),$$

and the general equation now takes the form:

$$\nabla^2 t = \left(\frac{\partial^2 t}{\partial \xi_1^2} + \frac{\partial^2 t}{\partial \xi_2^2}\right)\exp(-2\xi_1) + \frac{\partial^2 t}{\partial z^2} \qquad (14).$$

To pass to the ordinary cylindrical system it is convenient to take as radius and as azimuth of the functional point the following functions:

$$r = \exp(\xi_1)\ ;\ \varphi = \xi_2 \qquad (15),$$

and thus it is obtained:

$$\frac{\partial^2 t}{\partial \xi_1^2} = r^2 \frac{\partial^2 t}{\partial r^2} + r\frac{\partial t}{\partial r}\,;\ \frac{\partial^2 t}{\partial \xi_2^2} = \frac{\partial^2 t}{\partial \varphi^2} \quad (16),$$

and the Laplacian operator in the cylindrical coordinate system is given in this form:

$$\nabla^2 t = \frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \varphi^2} + \frac{\partial^2 t}{\partial z^2} \quad (17).$$

If the field does not depend on the z coordinate and the width or azimuth φ, then the Laplacian operator of the one-dimensional cylindrical field obtains the simplest form:

$$\nabla^2 t = \frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} \quad (18).$$

The Lame parameters for the spherical system are given according to:

$$H_1 = H_r = 1\,;\ H_2 = H_\varphi = r\cos\theta\,;\ H_3 = H_\theta = r \quad (19),$$

and the relevant Laplacian for this case is:

$$\nabla^2 t = \frac{\partial^2 t}{\partial r^2} + \frac{2}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2\cos^2\theta}\frac{\partial^2 t}{\partial \varphi^2} - \frac{tg\theta}{r^2}\frac{\partial t}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \theta^2} \quad (20).$$

If the process of heat transfer takes place in the spherical envelope with constant radius r, then the field depends only on the angular coordinates of θ width and the length φ. In this case the previous dependence is simplified:

$$\nabla^2 t = \frac{1}{r^2\cos^2\theta}\frac{\partial^2 t}{\partial \varphi^2} - \frac{tg\theta}{r^2}\frac{\partial t}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \theta^2} \quad (21).$$

If the field is symmetrical about the polar axis θ = $\pm\pi/2$, then the transmission of mass or energy is realized in the spherical layer or in the empty sphere, and the field does not depend on the length:

$$\nabla^2 t = \frac{\partial^2 t}{\partial r^2} + \frac{2}{r}\frac{\partial t}{\partial r} - \frac{tg\theta}{r^2}\frac{\partial t}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \theta^2} \quad (22),$$

but is given as a function of radius r and θ width.

In the case of a one-dimensional spherical field, then the previous dependence is simplified:

$$\nabla^2 t = \frac{\partial^2 t}{\partial r^2} + \frac{2}{r}\frac{\partial t}{\partial r} \quad (23).$$

2. Elaboration of some characteristic cases

In the special case, if the cylindrical or spherical field of temperature distribution is stationary, then it is valid:

$$\nabla^2 t \equiv \frac{d^2 t}{dr^2} + \frac{k-1}{r}\frac{dt}{dr} = 0 \qquad (24).$$

so that the parameter k = 2 for the cylindrical field and k = 3 for the spherical field.

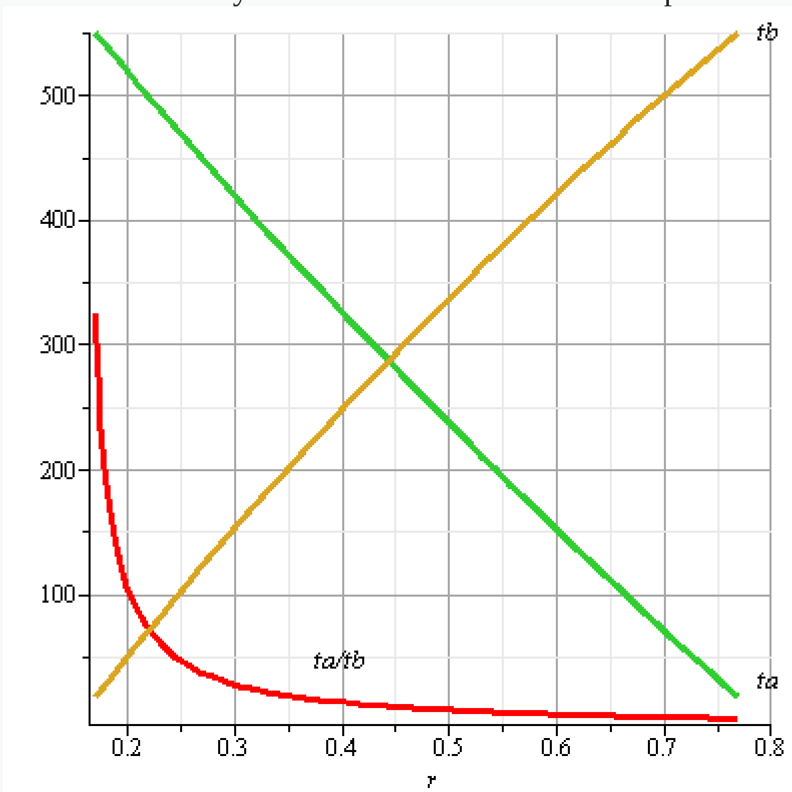

*Figure 1: Temperature difference for cylindrical and spherical wall***.**

Regarding the solution of the differential equation (24) for the cylindrical and the spherical wall, the diagram according to figure 1 is presented: For both walls the following design parameters are taken:

- r_1 = .17 (m) for internal radius;

- r_2 = 0.77 (m) for the outer radius of the wall;

- t_1 = 550 ^{0}C for the inner wall of the cylinder, respectively t_1 = 17 ^{0}C for the inner spherical wall;

- t_2 = 17 ^{0}C for the outer cylindrical wall (r_1 = .17 m), respectively t_2 = 550 ^{0}C for the outer spherical wall (r_2 = 0.77 m);

In figure 1 with t_a is given the change of temperature of the cylindrical wall, while with t_b - the change of temperature of the spherical wall.

For the Laplacian plane wall, for the stationary regime, is given according to the differential equation:

$$\nabla^2 t = \frac{d^2 t}{dx^2} = 0 \qquad (25).$$

By solving this differential equation we get:

$$t = t1 - \frac{t_1 - t_2}{\delta} x \qquad (26),$$

In connection with this mathematics, for stationary heat transmissions within the planar operational field, Laplacian can also relate to Fourier law, where the functional dependence of conduction heat transfer is given:

$$q = -\lambda \frac{dt}{dx} \qquad (27),$$

so that λ is the conduction coefficient, while q-is the thermal flux which passes through the single-walled wall with thickness δ.

If it is assumed that the conduction coefficient varies with temperature according to $\lambda = \lambda_0 (1 + bt)$, then according to (27) it is obtained:

$$t(x) = \frac{-\lambda_0 + \sqrt{\lambda_0^2 - 2\lambda_0 bqx + \lambda_0^2 bt_1 (2 + bt_1)}}{b} \qquad (28).$$

According to the dependence (28) is given the diagram in figure (2), and that for the design parameters: t_1 = 100 ^{0}C, λ_0 = 0.77 W / (mK), q = 1700 W / m^2.

It is noticed that with the increase of parameter b, the parametric curves with position from above in the diagram are obtained, so that the (nonlinear) trend of temperature change is smaller.

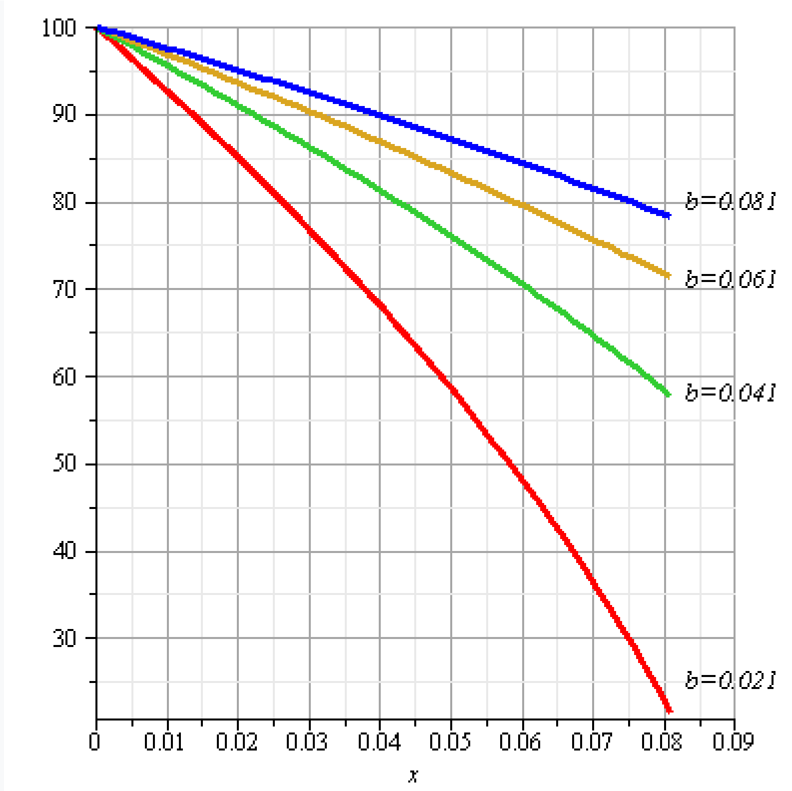

Figure 2: Temperature change when the conduction coefficient is a function of temperature.

Using the Laplacian operator to represent the one-dimensional field for the plate, cylinder or sphere, a general expression can be presented:

$$\nabla^2 t = \frac{d^2 t}{dr^2} + \frac{k-1}{r}\frac{dt}{dr} = r^{1-k}\frac{d}{dr}\left(r^{k-1}\frac{dt}{dr}\right) \tag{29}$$

3. Management statistical analysis for dedicated problematic

Regarding the managerial-statistical analysis of the problem in question, among others, it is more important to determine the size of the coefficient of elasticity. For each of the two variables (X, Y) the coefficient in question is determined according to:

$$\begin{gathered} \varepsilon(Y,X) = \frac{\partial Y}{\partial X}\frac{X}{Y} \\ \varepsilon(X,Y) = \frac{\partial X}{\partial Y}\frac{Y}{X} \\ \varepsilon(Y,X) + \varepsilon(X,Y) = 1 \end{gathered} \tag{30}$$

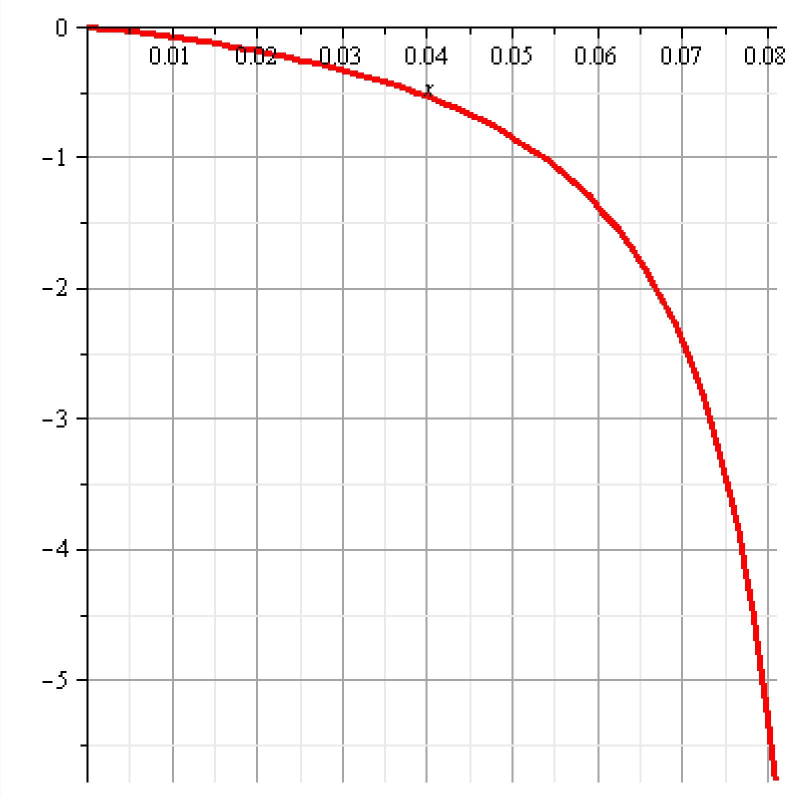

Figure 3: Elasticity coefficient ε(t,x) for treatment problematic.

Figure 3 graphically presents the coefficient of elasticity ε (t, x) according to the dependence (28). For example, for x = 0.4 is ε (t, x) = -0.5, which means that when the wall thickness of the plane changes by 1%, the temperature inside it decreases (minus sign) by 0.5%, and so on.

Discussion

The paper discusses the role of the Laplacian operator which has a comprehensive and very important use not only in thermotechnics and thermoenergy, but also in other scientific branches. Various cases have been elaborated, where in practice the examination of potential operational areas is of special importance, for the case when the temperature change is realized according to the stationary process. For non-stationary temperature change the preliminary analysis is more complicated. Relevant managerial-statistical analysis for the issue in question was also presented.

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