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Optimization problems

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Abstract

Optimization is a process that aims to find the best, most favorable, or most optimized solution for a given problem. This process includes the use of mathematical techniques, algorithms and specialized methods to identify the values of the variables that minimize, or maximize a certain function, which is called an objective function. The use of these methods helps to find valid and efficient solutions for optimization problems, bringing significant benefits in many areas of life and industry. The main goal of optimization is to identify the best or most favorable solution in the context of a given problem.

Optimization problems are present in many fields of life and sciences such as: engineering and design, transportation and logistics, artificial intelligence and machine learning, energy and natural resources, network management and telecommunications, sciences and environment, robotics and automation, economics , informatics, biology, statistics, finance, social sciences, genetic algorithms, metaheuristics, etc. To solve optimization problems, specialized optimization algorithms and methods are used, including browsing algorithms, linear programming, genetic algorithms, clustering methods and many others. etc. These algorithms help in finding valid and efficient solutions for various optimization problems.

In this paper, we will deal with some different problems from the real daily life of society. In solving such practical problems, the most important problem is often converting them into mathematical optimization problems by constructing a function that must be maximized or minimized.

Key words: optimization, minimization, maximization, variable, mathematical optimization.

Introduction

The methods that we will discuss in this paper for finding extremes find practical applications in many areas of life. A businessman seeks to minimize costs and maximize profits. A traveler seeks to minimize travel time. In this paper we will solve problems such as the maximization of surfaces, volumes, and benefits, as well as the minimization of distances, time, and cost.

In solving such practical problems, the most important problem is often converting them into mathematical optimization problems by constructing a function to be maximized or minimized.

Steps in solving optimization problems:

1. **Understand the problem**: Read the problem carefully to find out what the problem is asking. Then, underline the important pieces of information in the problem.
2. **Draw a diagram (optional):** It is always helpful to sum up the entire problem in a simple diagram so to prevent reading the problem repeatedly.
3. **Introduce notion and express it in terms of variables**: Assign a symbol to the quantity that has to be maximized or minimized. (Let us call it Z for now). Assign variable names to the unknowns and express Z as a function of those variables.
4. **Try to express notion in terms of one variable**: If Z has been expressed as a function of more than one variable (step 4), use the information in the problem to eliminate all but one of the variables and use that to express Z.
5. **Differentiate function and equate it to zero to obtain critical points:** Differentiate Z with respect to the variable you choose and equate it to zero to obtain the values of that variable (critical points).
6. **Test critical points for max/min using the second derivative**: To test whether the critical points are a max or min (concave down or up respectively), we take the second derivative of Z and plug in the critical points obtained in step 6 to see whether we get a positive value (minimum) or a negative value (maximum.)

7. **Use the required critical point to find the optimal answer**: Once we know what critical points we are using, we plug that in Z to obtain the answer to our problem.

Now let us use these steps on a few examples!

Example 1

A rectangular storage container with an open top needs to have a volume of 10 m^3 . The length of its base is twice the width. Material for the base costs $10 per m^2 . Material for the sides costs $6 per m^2 . Find the cost of the material for the cheapest container.

SOLUTION:

STEP 1: We have:

a) Rectangular box with open top (**info**)

b) Volume of the box is 10 m^3 (**info**)

c) Length of the base is twice its width (**info**)

d) Material for the base costs $10 / m^2 and material for the sides costs $6 / m^2 (**info**)

e) The cost of the material for the **cheapest** container (**objective**)

STEP 2:

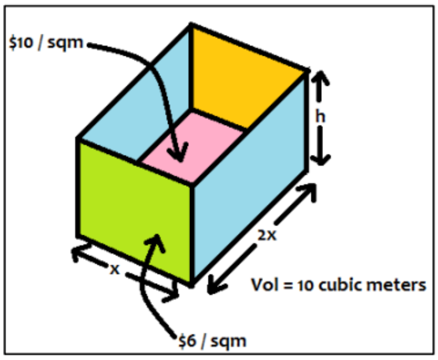

Fig.1

STEP 3: Let us assume the height of the box is **h** and the width of the box is **x**. Then from the information given, we obtain the length of the box as **2x**. Let the cost of the box be **C**.

Cost = $10(area of base) + $6(area of 2 long sides) + $6(area of 2 short sides) which gives,

$$Cost = (10 \cdot x \cdot 2x) + (2 \cdot 6 \cdot x \cdot h) + (2 \cdot 6 \cdot 2x \cdot h) = 20x^2 + 12 \cdot x \cdot h + 24 \cdot x \cdot h \ = 20x^2 + 36 \cdot x \cdot h$$

STEP 4: Since **C** is expressed in terms of both **x** and **h**, we need to try to express **S** with only one variable. From the information given, we have the volume of the box = 10m^3 , which means:

$x \cdot 2x \cdot h = 10$ or $h = \dfrac{10}{2x^2}$.

Putting this value of h in **C** we get:

$C(x) = 20x^2 + \dfrac{36x10}{2x^2}$ .When we simplify this, $C(x) = 20x^2 + \dfrac{180}{x}$, which we need to **minimize**.

Example 2. A farmer has 2400m of wire and wants to fence a rectangular field which is bounded on one side by a river. He wants to fence off the part of the field shown in the picture.

What are the dimensions of the field so that it has the largest view?

solution
We would like to maximize the function **S** of the straight line. Let **x** and **y** be the latitude and longitude of the ruler (in meters). Then, express **S** in terms of **x** and **y**:

$$S = x \cdot y$$

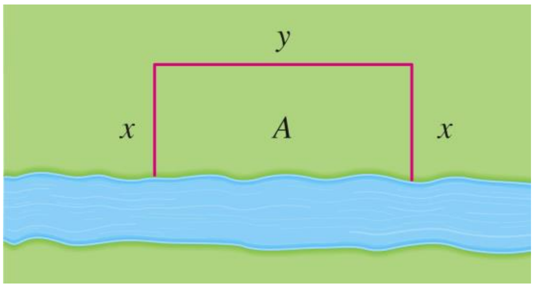

Fig.2

We want to express **S** as a function of only one variable, so we eliminate **y** by expressing it as a function of **x**. To do this, we use the information given that the total length is **2400m**, thus:

2x + y = 2400

From this equation we have **y = 2400 - 2x**, which gives the expression:

$$S = x \cdot (2400 - 2x)$$

$$S = 2400x - 2x^2$$

We note that $x \geq 0$ and $x \leq 1200$.(otherwise it would be **S < 0**).

So the function we want to maximize is:

$S(x) = 2400x - 2x^2$ and $0 \leq x \leq 1200$

Derivative of $S(x) = 2400x - 2x^2$, will be:

$S'(x) = 2400 - 4x$, therefore to find the critical points we solve the equation:

2400 - 4x = 0
whose solution is: **x = 600**.

The maximum value of **S** can occur either at the **critical point** or at the **edges of the interval**.

Since **S(0) = 0**, **S(600) = 720,000**, and **S(1200) = 0**, the closed interval method gives the maximum value

S(600) = 720,000.

So, the rectangular field should be **600m wide** and **1200m long**.

Example 3. A cylindrical container is made to hold 1L of oil. Find the dimensions that minimize the cost of the metal needed to make the oil container.

Solution: Let **r** be the radius and **h** the height of the cylinder (both in centimeters). In order to minimize the metal cost, we minimize the total surface area of the cylinder. The lateral surface is a rectangle with dimensions: $2\pi r$ and **h ,** Therefore the surface is:

$S = 2\pi r^2 + 2\pi rh$ and $V = \pi r^2 h$, to eliminate use the fact that the volume is 1L, which falls to be: **V=1000cm^3**. Therefore:

$\pi r^2 h = 1000$, and from here: $h = \dfrac{1000}{\pi r^2}$

By substituting this in the expression of syprina we will have:

$$S = 2\pi r^2 + 2\pi r \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}$$

Then the function we want to minimize is:

$$S(r) = 2\pi r^2 + \frac{2000}{r}, r > 0$$

To find the critical point, we derive the expression: $S(r) = 2\pi r^2 + \dfrac{2000}{r}$ and we will have:

Then: $S'(r) = 0$ vhen $\pi r^3 - 500$ =0, therefore the only critical point is: $r = \sqrt[3]{\dfrac{500}{\pi}}$

Since the definition set of S is $(0,\infty)$, we can notice that $S'(r) < 0$ for $r < \sqrt[3]{\dfrac{500}{\pi}}$

And $S'(r) > 0$ for $r > \sqrt[3]{\dfrac{500}{\pi}}$

So $r = \sqrt[3]{\dfrac{500}{\pi}}$ constitutes the local minimum point.

The corresponding value of **h** with respect to **r** is:

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = \frac{1000}{\pi\left(\frac{500}{\pi}\right)^{\frac{2}{3}}} = 2\sqrt[3]{\frac{500}{\pi}} = 2\cdot r$$

Therefore to minimize the production cost of the cylindrical container the radius of the base should be: $r = \sqrt[3]{\dfrac{500}{\pi}}$ and the height twice the radius **(h=2r).**

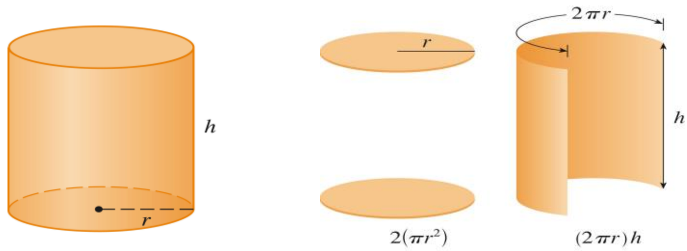

Fig.3

Example 4. Find the point of the parabola $y^2 = 2x$ which is closer to the point **(1;4).**

Solution: The distance between point **(1;4)** and **(x;y)** is:

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

But if (x; y) is located on the parabola, then $x = \frac{y^2}{2}$, so the expression for **d** becomes:

$$d = \sqrt{\left(\frac{1}{2}y^2 - 1\right)^2 + (y-4)^2}$$

(or else we could substitute $y = \sqrt{2x}$ to derive d in terms of **x** only)

Instead of minimizing **d** we minimize : d^2

$$d^2 = f(y) = \left(\frac{1}{2}y^2 - 1\right)^2 + (y-4)^2$$

Deriving we have:

$$f'(y) = 2\left(\frac{1}{2}y^2 - 1\right)y + 2(y-4) = y^3 - 8$$

thus $f'(y) = 0$,**y=2** .We note that $f'(y) < 0$ when **y < 2** and $f'(y) > 0$ vhen **y>2**, so by the first derivative test and the absolute extreme value test, the absolute minimum occurs for **y = 2**. The corresponding value of x is: $x = \frac{y^2}{2} = 2$.

So the point on the parabola $y^2 = 2x$ that is closest to the point **(1;4)** is **(2;2).**

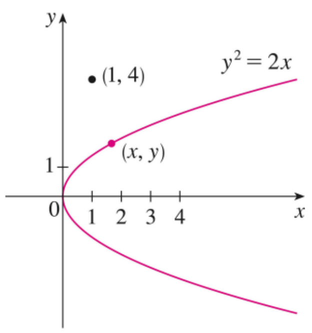

Fig.4

Example 5. A man launched his boat on a river bank **3 km** wide at point **A** and wants to reach point **B**, **8 km** away from the corresponding point of **A** on the other side of the river, as quickly as possible. He can steer the boat straight to point **C** and then go to **B**, but he can go directly to point **B**, but he can also go first to a point **D** between **C** and **B** and from there to **B**. If that he drives at **6km=h** and runs at **8km=h**, where should he go to reach **B** as fast as possible? (We admit that the speed of the river is negligible compared to the speed at which a person paddles.)

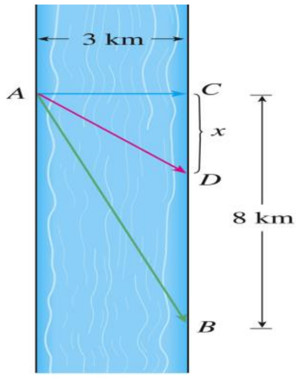

Fig.5

Solution:

Let **x** be the distance from **C** to **D**. Then the running distance is: $|DB| = 8 - x$, and the sailing distance is: $|AD| = \sqrt{x^2 + 9}$ Then the sailing time is: $\frac{\sqrt{x^2+9}}{6}$ and running time: $\frac{8-x}{8}$ and the total time as a function of **x** is:

$$T(x) = \frac{\sqrt{x^2+9}}{6} + \frac{8-x}{8}$$

The definition set of the function **T** is **[0;8].** Note that if **x = 0** he drives to **C** and if x = 8 he drives directly to **B**. The derivative of **T** is:

$$T'(x) = \frac{x}{6\sqrt{x^2+9}} - \frac{1}{8}$$

Therefore using the fact that $x \geq 0$ have $T'(x) = 0$, it follows that:

$$\frac{x}{6\sqrt{x^2+9}} - \frac{1}{8} \Rightarrow 4x = 3\sqrt{x^2+9} \Rightarrow$$

$$\Rightarrow 16x^2 = 9\left(x^2+9\right) \Rightarrow x = \frac{9}{\sqrt{7}}$$

The only critical point is $x = \frac{9}{\sqrt{7}}$. To see where the minimum is located, at the critical point or at the edges of the definition set **[0;8]** ,we calculate T at all three points from the expression:

$T(x) = \frac{\sqrt{x^2+9}}{6} + \frac{8-x}{8}$, and we will have:

$$T(0) = 1.5$$

$$T\left(\frac{9}{\sqrt{7}}\right) = 1 + \frac{\sqrt{7}}{8} \approx 1.33$$

$$T(8)=\frac{\sqrt{73}}{6}\approx 1.42$$

Since the smallest value of these values is located for $x=\frac{9}{\sqrt{7}}$ then the absolute minimum is located right here.

Example 6. Find the area of the largest rectangle that can be inscribed in a semicircle with radius **r**.

Solution: Let be the semicircle of the circle with radius **r**, $x^2+y^2=r^2$ centered at the origin of the coordinates. The inscribed word means that the rectangle has two vertices on the semicircle and two more vertices on the **x**-axis. Let **(x;y)** be a vertex located in the first quadrant. Then, the rectangle has sides of length **2x** and **y**, so its surface will be $S=2\cdot x\cdot y$.

To eliminate **y** we use the fact that **(x;y)** is on the circle $x^2+y^2=r^2$ and from here: $y=\sqrt{r^2-x^2}$

$S=2xy=2x\sqrt{r^2-x^2}$, The set of definitions of this function is $0\le x\le r$ also the derivative:

$$S'(r)=2\sqrt{r^2-x^2}-\frac{2x^2}{\sqrt{r^2-x^2}}=\frac{2\left(r^2-2x^2\right)}{\sqrt{r^2-x^2}}$$, which is zero when:

$2x^2=r^2$ thus $x=\frac{r}{\sqrt{2}}$

This value of **x** gives the maximum value of **S** since **S(0) = 0** and **S(r) = 0**.

Since the surface of the largest rectangle inscribed in the semicircle is:

$$S\left(\frac{r}{\sqrt{2}}\right)=2\frac{r}{\sqrt{2}}\sqrt{r^2-\frac{r^2}{2}}=2\frac{r}{\sqrt{2}}\sqrt{\frac{2r^2-r^2}{2}}=2\frac{r}{\sqrt{2}}\sqrt{\frac{r^2}{2}}=2\frac{r}{\sqrt{2}}\cdot\frac{r}{\sqrt{2}}=r^2$$

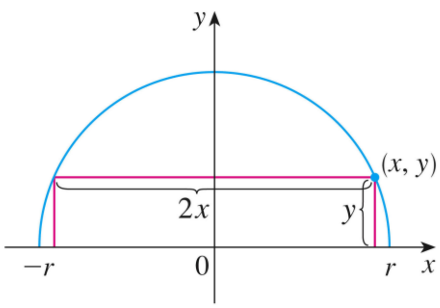

Fig.6

Example 7 Calculate the arc length of the given curve with the function:

$$f(x)=\sqrt{x-x^2}+\arcsin\sqrt{x}$$

Solution:

Since the limits of integration are not given, we first look for the area of definition of the function:

$$\left.\begin{array}{l} x-x^2 \geq 0 \\ x \geq 0 \end{array}\right\} \Rightarrow x \in [0,1] \ and \ \ f'(x) = \frac{1-x}{\sqrt{x}\sqrt{1-x}} \cdot \frac{\sqrt{1-x}}{\sqrt{1-x}} = \frac{\sqrt{1-x}}{\sqrt{x}} = \sqrt{\frac{1-x}{x}}$$

Therefore, the arc length of the curve will be:

$$L = \int_0^1 \sqrt{1+[f'(x)]^2}dx = \int_0^1 \sqrt{1+\frac{1-x}{x}}dx = \int_0^1 \sqrt{\frac{1}{x}}dx = 2\sqrt{x}\Big|_0^1 = 2$$

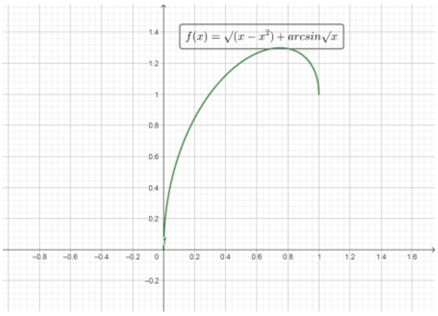

Fig.7

Example 8 *Consider a ball thrown vertically upwards from the ground. The ball position at each instant follows the function:* $h(t) = 40 \cdot t - 5t^2$, *where **h** is expressed in meters and **t** in seconds.*
*Determine: **(a)** The height at which the ball lies **1 second** after the throw; **(b)** The instant at which it is **75m** from the ground; **(c)** The maximum height reached by the ball and **(d)** The instant when the ball returns to the ground.*

Solution

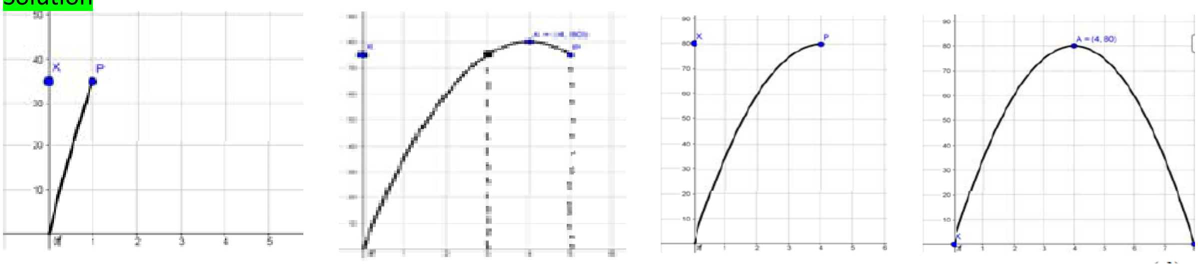

Fig.8

a) To find the height of the ball after the first second, we act in this way: we replace the value **t=1** in the above equation and we will have: $h(1) = 40 \cdot 1 - 5 \cdot 1^2 = 40 - 5 = 35m$.

b) To find the answer after how long will the ball reach the height of **75 meters**, we act in this way:

$$h(t) = 40 \cdot t - 5t^2 = 75 / :5$$

$$-t^2 + 8t - 15 = 0 / \cdot(-1)$$

$$t^2 - 8t + 15 = 0 \Rightarrow (t-3)(t-5) = 0 \Rightarrow t_1 = 3 \ and \ t_2 = 5$$

c) The maximum height the ball will reach for the time: $t = -\frac{b}{2 \cdot a} = -\frac{(-8)}{2 \cdot 1} = \frac{8}{2} = 4\sec$.

The max height of the ball is:

$$h(4) = 40 \cdot 4 - 5 \cdot 4^2 = 160 - 5 \cdot 16 = 160 - 80 = 80m$$

d) The ball will touch the ground again for time t=8 sec.

Example 9 The profit (in thousands of dollars) of a company is given by: $P(x) = 5000 + 1000x - 5x^2$

where **x** is the amount (in thousands of dollars) the company spends on advertising.

1. Find the amount, **x**, that the company has to spend to maximize its profit.

2. Find the maximum profit **Pmax**.

Zgjidhje

a. Function **P** that gives the profit is a quadratic function with the leading coefficient **a = -5**.

This function (profit) has a maximum value at: $x = h = -\frac{b}{2a}$

$$x = h = -\frac{b}{2a} = -\frac{1000}{2\cdot(-5)} = -\frac{1000}{-10} = 100$$

b. The maximum profit **Pmax**, when **x = 100** thousands is spent on advertising, is given by the maximum value of **function P**

$$k = c - \frac{b^2}{4a}$$

c. The maximum profit **Pmax**, when **x = 100** thousands is spent on advertising, is also given by **P(h = 100)**

$$P(100) = 5000 + 1000(100) - 5(100)^2 = 55000.$$

d. When the company spends **100** thousands dollars on advertising, the profit is maximum and equals **55000** dollars.

e. Shown below is the graph of **P(x),** notice the maximum point, vertex, at **(100 , 55000)**.

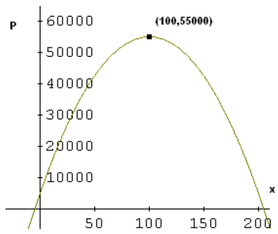

Fig.9

Example 10 An object is thrown vertically upward with an initial velocity of **V_o** feet/sec. Its distance **S(t)**, in feet, above ground is given by : $S(t) = -16t^2 + v_0 t$

Find $v_0 = ?$,so that the highest point the object can reach is **300 feet above ground**.

Zgjidhje

a. **S(t)** is a quadratic function and the maximum value of **S(t)** is given by :

$$k = c - \frac{b^2}{4a} = 0 - \frac{v_0^{\ 2}}{4\cdot(-16)}$$

b. This maximum value of **S(t)** has to be **300** feet in order for the object to reach a maximum distance above ground of **300** feet.

$$-\frac{v_0^{\ 2}}{4\cdot(-16)}=300$$

c. we now solve $-\frac{v_0^{\ 2}}{4\cdot(-16)}=300$ for **v₀**

$$\frac{v_0^{\ 2}}{4\cdot 16}=300 \Rightarrow v_0^{\ 2}=64\cdot 300 \Rightarrow v_0^{\ 2}=19200 \Rightarrow v_0=\sqrt{19200} \Rightarrow v_0=138.564$$ **feet/sec**.

The graph of S(t) for $v_0=138.564$ **feet/sec** is shown below.

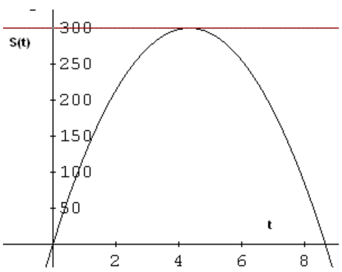

Fig.10

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